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A
SUPPLEMENT
TO THE
DOCTRINE
OF
ANNUITIES and REVERSIONS,
DEDUCED FROM
GENERAL and EVIDENT PRINCIPLES:
CONTAINING THE VALUATION OF
ANNUITIES and REVERSIONS,
FOR
SINGLE AND JOINT LIVES;
With the necessary TABLES.

By THOMAS SIMPSON, F. R. S.

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P R E F A C E.

THE following Treatise was first published by Mr. SIMPSON in his *Select Exercises for young Proficients in the Mathematicks* in the year 1752, and is now reprinted, (together with the Chapter on Interest and Annuities, which forms a part of his *Algebra*,) with the view of uniting together all that the learned author has written on this subject. Though it contains many problems which are solved in the foregoing Treatise, it has been thought best not to omit any of them, but to refer to the pages in which the corresponding problems may be found. This is the more necessary, not only that the work might be preserved entire, but also as some of the solutions are derived from more accurate principles, and the rules are illustrated by different and more general examples.

The 3d, 4th, and 5th tables in the former edition, being the same with the two tables in pag. 112 and 114, and the 1st table in the *Doctrine of Annuities*, are now omitted, and such alterations made in the references as became necessary in consequence of those omissions.

The AUTHOR'S ADVERTISEMENT.

THIS Second Part is concerning the Valuation of Annuities and Reversions, for single and joint Lives; wherein, besides the necessary tables, are given the solutions of upwards of forty different problems, on the most important and intricate cases of the subject: many of which are quite new, and are, besides, such as actually occur in business, being, most of them, taken from real cases, proposed to the author's consideration, by gentlemen in the law, and others. The reader will perceive it was designed as a Supplement to my Doctrine of Annuities and Reversions, printed in 1742. As to the performance itself, I do not in the least doubt but that it will be depreciated by some, on account of the observations whereon the calculations are grounded. I am sensible that there neither is, nor can be, a table of observations on the degrees of mortality of mankind, but what may be objected to; and that those persons who make a very despicable figure, when they come to calculations, seldom fail of displaying their talents, and being illustrious here; where, gratifying themselves in the liberty which the nature of the subject allows them, they can boldly launch out, without having to do with science and demonstration. But, though the London Bills of Mortality, whereon I build, appear to me to be the best foundation, at least, for this place; yet I have no inclination to enter the lists with any of these gentlemen. The examples, given hereafter, are indeed wrought according to the London Bills; but the solutions themselves are general, and may be applied, with equal facility and advantage, to any table of Observations.

May, 1752.

The

The VALUATION of ANNUITIES and
REVERSIONS, for single and joint
LIVES.

THE following Tract being designed for *real use*, and adapted to the understanding of such whose acquisitions, in the Mathematical way, extend not beyond vulgar, or decimal arithmetick; it therefore seemed proper to omit the investigation of several particulars therein delivered, depending upon higher principles. This, I hope, my mathematical readers will have the candour to excuse; when they consider the importance of the subject to a multitude of persons, who cannot be expected to see into the nature and usefulness of an algebraical process.—Without further apology I shall therefore proceed now to my purpose; which is to exhibit, in a plain, easy manner, by means of proper tables, the practical solutions of the most useful and necessary problems on the subject; without any intermixture of *analytical operations* (whereof the bare appearance, to those unacquainted therewith, would seem to cast a darkness over the *whole*). However, for the sake of those that are judges, I shall, in an annexed Scholium, give the reasons of what is most material, and necessary to be explained.

TABLE

6 Of the Values of Annuities and Reversions,

TABLE VI.

Shewing the Probability of the Duration of Life, from Observations on the Bills of Mortality of the City of London.

Years.	N ^o . Persons.	Years.	N ^o . Persons.	Years.	N ^o . Persons.	Years.	N ^o . Persons.
0	1000	20	360	40	229	60	102
1	680	21	355	41	222	61	97
2	547	22	350	42	214	62	92
3	496	23	345	43	206	63	87
4	469	24	339	44	199	64	82
5	452	25	333	45	192	65	77
6	440	26	327	46	185	66	72
7	430	27	321	47	178	67	67
8	422	28	315	48	171	68	62
9	415	29	308	49	165	69	58
10	410	30	301	50	159	70	54
11	405	31	294	51	153	71	50
12	400	32	287	52	147	72	46
13	395	33	280	53	141	73	42
14	390	34	273	54	135	74	39
15	385	35	266	55	129	75	36
16	380	36	259	56	123	76	33
17	375	37	252	57	117	77	30
18	370	38	245	58	112	78	27
19	365	39	237	59	107	79	25

TABLE VII.

Exhibiting the Number of Years of Life, which a Person, of a given Age, may, upon an Equality of Chance, expect to enjoy; according to the aforesaid Observations.

Age.	Exp.	Age.	Exp.	Age.	Exp.	Age.	Exp.
1	27.0	21	28.3	41	19.2	61	12.6
2	32.0	22	27.7	42	18.8	62	11.6
3	34.0	23	27.2	43	18.5	63	11.2
4	35.6	24	26.6	44	18.1	64	10.8
5	36.0	25	26.1	45	17.8	65	10.5
6	36.0	26	25.6	46	17.4	66	10.1
7	35.8	27	25.1	47	17.0	67	9.8
8	35.6	28	24.6	48	16.7	68	9.4
9	35.2	29	24.1	49	16.3	69	9.1
10	34.8	30	23.6	50	16.0	70	8.8
11	34.3	31	23.1	51	15.6	71	8.4
12	33.7	32	22.7	52	15.2	72	8.1
13	33.1	33	22.3	53	14.9	73	7.8
14	32.5	34	21.9	54	14.5	74	7.5
15	31.9	35	21.5	55	14.2	75	7.2
16	31.3	36	21.1	56	13.8	76	6.8
17	30.7	37	20.7	57	13.4	77	6.4
18	30.1	38	20.3	58	13.1	78	6.0
19	29.5	39	19.9	59	12.7	79	5.5
20	28.9	40	19.6	60	12.4	80	5.0

8 Of the Values of Annuities and Reversions,

TABLE VIII.

Exhibiting the Value of an Annuity for a given Term of Years, on the Contingency of its ceasing upon the Extinction of an assigned Life.

Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
5	5	4.0	4.1	4.2	15	5	4.1	4.2	4.4
	10	7.0	7.3	7.7		10	7.1	7.5	7.9
	15	9.1	9.8	10.5		15	9.3	10.0	10.7
	20	10.7	11.7	12.8		20	10.8	11.8	12.9
	25	11.8	13.0	14.5		25	11.8	13.1	14.5
	30	12.6	14.1	15.8		30	12.5	14.0	15.7
	35	13.2	14.9	16.8		35	13.0	14.6	16.6
	40	13.6	15.4	17.5		40	13.3	15.0	17.2
	45	13.8	15.7	18.0		45	13.5	15.3	17.6
	50	13.9	15.9	18.3		50	13.6	15.5	17.9
10	5	4.1	4.2	4.3	20	5	4.1	4.2	4.3
	10	7.2	7.6	8.0		10	7.0	7.4	7.8
	15	9.4	10.1	10.8		15	9.1	9.7	10.5
	20	11.0	12.0	13.1		20	10.5	11.4	12.5
	25	12.1	13.4	14.8		25	11.4	12.6	13.9
	30	12.8	14.4	16.2		30	12.0	13.4	15.0
	35	13.3	15.1	17.1		35	12.5	13.9	15.7
	40	13.7	15.5	17.7		40	12.7	14.3	16.3
	45	13.9	15.8	18.2		45	12.8	14.5	16.6
	50	14.0	16.0	18.5		50	12.9	14.6	16.8

for single and joint LIVES.

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Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
25	5	4.1	4.2	4.3
	10	6.9	7.3	7.6
	15	8.9	9.5	10.1
	20	10.2	11.0	12.0
	25	11.0	12.1	13.4
	30	11.5	12.8	14.3
	35	11.8	13.2	15.0
	40	12.0	13.5	15.4
30	5	4.0	4.1	4.2
	10	6.8	7.1	7.5
	15	8.6	9.2	9.8
	20	9.8	10.6	11.5
	25	10.6	11.6	12.7
	30	11.0	12.2	13.5
	35	11.2	12.6	14.0
	40	11.4	12.8	14.4
35	5	4.0	4.1	4.2
	10	6.6	6.9	7.3
	15	8.3	8.9	9.5
	20	9.4	10.2	11.1
	25	10.1	11.1	12.2
	30	10.5	11.6	13.0
	35	10.7	11.9	13.5
40	5	3.9	4.0	4.1
	10	6.4	6.8	7.1
	15	8.0	8.6	9.2
	20	9.1	9.8	10.6
	25	9.7	10.6	11.6
	30	10.0	11.0	12.2
45	5	3.9	4.0	4.1
	10	6.3	6.7	7.0
	15	7.9	8.4	9.0
	20	8.8	9.5	10.3
	25	9.4	10.2	11.2
	30	9.6	10.6	11.7
50	5	3.8	3.9	4.0
	10	6.2	6.5	6.8
	15	7.6	8.2	8.8
	20	8.5	9.1	9.9
	25	8.9	9.7	10.6
	5	3.8	3.9	4.0
	10	6.1	6.4	6.7
	15	7.4	7.9	8.4
55	5	3.8	3.9	4.0
	10	6.1	6.4	6.7
	15	7.4	7.9	8.4
	20	8.0	8.7	9.4
	25	8.3	9.0	9.9
60	5	3.7	3.8	3.9
	10	5.8	6.1	6.4
	15	7.0	7.4	7.9
	20	7.5	8.0	8.7
	5	3.6	3.7	3.8
	10	5.4	5.7	6.0
	15	6.4	6.8	7.2
65	5	3.4	3.5	3.6
	10	5.0	5.2	5.4
	5	3.4	3.5	3.6
	10	5.0	5.2	5.4
	5	3.4	3.5	3.6
	10	5.0	5.2	5.4
	5	3.4	3.5	3.6
	10	5.0	5.2	5.4

C

TABLE

10 *Of the Values of Annuities and Reversions,*

TABLE IX.

Serving as a Supplement to TABLE VIII.

Diff.	1	2	3	4
3.4	0.8	1.6	2.4	2.9
3.3	0.8	1.5	2.2	2.8
3.2	0.8	1.5	2.1	2.7
3.1	0.7	1.4	2.0	2.6
3.0	0.7	1.3	1.9	2.5
2.9	0.7	1.3	1.9	2.4
2.8	0.7	1.3	1.8	2.3
2.7	0.6	1.2	1.7	2.2
2.6	0.6	1.2	1.7	2.2
2.5	0.6	1.1	1.6	2.1
2.4	0.6	1.1	1.6	2.0
2.3	0.5	1.0	1.5	1.9
2.2	0.5	1.0	1.4	1.8
2.1	0.5	0.9	1.3	1.7
2.0	0.5	0.9	1.3	1.7
1.9	0.4	0.8	1.2	1.6
1.8	0.4	0.8	1.1	1.5
1.7	0.4	0.7	1.1	1.4
1.6	0.4	0.7	1.0	1.3
1.5	0.4	0.7	1.0	1.3
1.4	0.3	0.6	0.9	1.2
1.3	0.3	0.6	0.9	1.1
1.2	0.3	0.6	0.8	1.0
1.1	0.3	0.5	0.7	0.9
1.0	0.2	0.4	0.6	0.8

TABLE

10

12 Of the Values of Annuities and Reversions.

TABLE X.

Shewing the Value of an Annuity for two joint Lives
(i. e. for as long as They exist together.)

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
10	10	11.6	13.0	14.7	20	20	10.1	11.3	12.8
	15	11.3	12.7	14.3		25	9.7	10.8	12.2
	20	10.8	12.2	13.8		30	9.2	10.3	11.6
	25	10.2	11.6	13.1		35	8.8	9.8	10.9
	30	9.7	10.9	12.3		40	8.4	9.2	10.2
	35	9.1	10.2	11.5		45	7.9	8.6	9.5
	40	8.6	9.6	10.7		50	7.4	8.0	8.8
	45	8.1	9.0	10.0		55	6.9	7.5	8.1
	50	7.6	8.4	9.3		60	6.4	6.9	7.4
	55	7.1	7.8	8.6		65	5.9	6.3	6.7
	60	6.6	7.2	7.8		70	5.4	5.7	6.0
	65	6.1	6.5	6.9		75	4.8	5.0	5.2
	70	5.5	5.8	6.1		25	9.4	10.5	11.8
	75	4.9	5.1	5.3		30	9.0	10.1	11.3
15	15	11.0	12.3	13.9	25	35	8.6	9.6	10.7
	20	10.5	11.8	13.3		40	8.2	9.1	10.0
	25	10.1	11.2	12.6		45	7.8	8.5	9.4
	30	9.5	10.6	11.9		50	7.3	7.9	8.7
	35	9.0	10.0	11.2		55	6.8	7.4	8.0
	40	8.5	9.4	10.4		60	6.3	6.8	7.3
	45	8.0	8.8	9.6		65	5.8	6.2	6.6
	50	7.5	8.2	8.9		70	5.3	5.6	5.9
	55	7.0	7.6	8.2		75	4.7	4.9	5.1
	60	6.5	7.0	7.5	30	30	8.6	9.6	10.8
	65	6.0	6.4	6.8		35	8.3	9.2	10.3
	70	5.4	5.7	6.0		40	8.0	8.8	9.7
	75	4.8	5.0	5.2					

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
30	45	7.6	8.3	9.1	45	65	5.4	5.8	6.3
	50	7.2	7.8	8.5		70	5.0	5.3	5.6
						75	4.5	4.7	4.9
	55	6.7	7.3	7.9	50	50	6.2	6.8	7.6
	60	6.2	6.7	7.2		55	6.0	6.5	7.2
	65	5.7	6.1	6.5		60	5.7	6.1	6.7
35	70	5.2	5.5	5.8		65	5.3	5.7	6.2
	75	4.7	4.9	5.1		70	4.9	5.2	5.5
						75	4.4	4.6	4.8
	35	8.0	8.8	9.9	55	55	5.7	6.2	6.9
	40	7.7	8.5	9.4		60	5.5	5.9	6.5
	45	7.4	8.1	8.9		65	5.2	5.6	6.0
	50	7.0	7.6	8.3		70	4.8	5.1	5.4
	55	6.6	7.1	7.7		75	4.3	4.5	4.7
	60	6.1	6.5	7.1	60	60	5.2	5.6	6.1
40	65	5.6	6.0	6.4		65	4.9	5.3	5.7
	70	5.1	5.4	5.7		70	4.6	4.9	5.2
	75	4.6	4.8	5.0		75	4.2	4.4	4.6
					65	65	4.7	5.0	5.4
	40	7.3	8.1	9.1		70	4.4	4.6	4.9
	45	7.1	7.8	8.7		75	4.0	4.2	4.4
45	50	6.8	7.4	8.2	70	70	4.2	4.4	4.6
	55	6.4	6.9	7.6		75	3.9	4.0	4.2
	60	6.0	6.4	7.0	75	75	3.6	3.7	3.8
	65	5.5	5.9	6.4					
	70	5.1	5.4	5.7					
	75	4.6	4.8	5.0					

TABLE

12 Of the Values of Annuities and Reversions.

TABLE X.

*Shewing the Value of an Annuity for two joint Lives
(i. e. for as long as They exist together.)*

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
10	10	11.6	13.0	14.7	20	20	10.1	11.3	12.8
	15	11.3	12.7	14.3		25	9.7	10.8	12.2
	20	10.8	12.2	13.8		30	9.2	10.3	11.6
	25	10.2	11.6	13.1		35	8.8	9.8	10.9
	30	9.7	10.9	12.3		40	8.4	9.2	10.2
	35	9.1	10.2	11.5		45	7.9	8.6	9.5
	40	8.6	9.6	10.7		50	7.4	8.0	8.8
	45	8.1	9.0	10.0		55	6.9	7.5	8.1
	50	7.6	8.4	9.3		60	6.4	6.9	7.4
	55	7.1	7.8	8.6		65	5.9	6.3	6.7
	60	6.6	7.2	7.8		70	5.4	5.7	6.0
	65	6.1	6.5	6.9		75	4.8	5.0	5.2
	70	5.5	5.8	6.1		25	9.4	10.5	11.8
	75	4.9	5.1	5.3		30	9.0	10.1	11.3
15	15	11.0	12.3	13.9	25	35	8.6	9.6	10.7
	20	10.5	11.8	13.3		40	8.2	9.1	10.0
	25	10.1	11.2	12.6		45	7.8	8.5	9.4
	30	9.5	10.6	11.9		50	7.3	7.9	8.7
	35	9.0	10.0	11.2		55	6.8	7.4	8.0
	40	8.5	9.4	10.4		60	6.3	6.8	7.3
	45	8.0	8.8	9.6		65	5.8	6.2	6.6
	50	7.5	8.2	8.9		70	5.3	5.6	5.9
	55	7.0	7.6	8.2		75	4.7	4.9	5.1
	60	6.5	7.0	7.5	30	30	8.6	9.6	10.8
	65	6.0	6.4	6.8		35	8.3	9.2	10.3
	70	5.4	5.7	6.0		40	8.0	8.8	9.7
	75	4.8	5.0	5.2					

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
30	45	7.6	8.3	9.1	45	65	5.4	5.8	6.3
	50	7.2	7.8	8.5		70	5.0	5.3	5.6
						75	4.5	4.7	4.9
	55	6.7	7.3	7.9	50	50	6.2	6.8	7.6
	60	6.2	6.7	7.2		55	6.0	6.5	7.2
	65	5.7	6.1	6.5		60	5.7	6.1	6.7
	70	5.2	5.5	5.8		65	5.3	5.7	6.2
	75	4.7	4.9	5.1		70	4.9	5.2	5.5
						75	4.4	4.6	4.8
						55	5.7	6.2	6.9
						60	5.5	5.9	6.5
						65	5.2	5.6	6.0
35	35	8.0	8.8	9.9		70	4.8	5.1	5.4
	40	7.7	8.5	9.4		75	4.3	4.5	4.7
	45	7.4	8.1	8.9	60	60	5.2	5.6	6.1
	50	7.0	7.6	8.3		65	4.9	5.3	5.7
	55	6.6	7.1	7.7		70	4.6	4.9	5.2
	60	6.1	6.5	7.1		75	4.2	4.4	4.6
	65	5.6	6.0	6.4	65	65	4.7	5.0	5.4
	70	5.1	5.4	5.7		70	4.4	4.6	4.9
	75	4.6	4.8	5.0		75	4.0	4.2	4.4
					70	70	4.2	4.4	4.6
40	40	7.3	8.1	9.1		75	3.9	4.0	4.2
	45	7.1	7.8	8.7	75	75	3.6	3.7	3.8
	50	6.8	7.4	8.2					
	55	6.4	6.9	7.6					
	60	6.0	6.4	7.0					
	65	5.5	5.9	6.4					
	70	5.1	5.4	5.7					
	75	4.6	4.8	5.0					
45	45	6.7	7.4	8.3					
	50	6.5	7.1	7.9					
	55	6.2	6.7	7.4					
	60	5.8	6.3	6.8					

TABLE

14 *Of the Values of Annuities and Reversions.*

TABLE XI.

For the Value of an Annuity upon the longest of two given Lives.

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
10	10	17.1	19.9	23.4	20	20	15.8	18.3	21.6
	15	16.8	19.5	22.9		25	15.5	17.9	21.1
	20	16.6	19.1	22.5		30	15.3	17.6	20.7
	25	16.4	18.8	22.2		35	15.1	17.4	20.4
	30	16.2	18.6	21.9		40	15.0	17.2	20.1
	35	16.1	18.4	21.6		45	14.9	17.0	19.9
	40	16.0	18.3	21.4		50	14.7	16.8	19.6
	45	15.9	18.2	21.2		55	14.5	16.6	19.4
	50	15.8	18.0	20.9		60	14.3	16.3	19.1
	55	15.7	17.8	20.7		65	14.1	16.0	18.7
	60	15.5	17.6	20.4		70	13.8	15.7	18.2
	65	15.3	17.4	20.1		75	13.5	15.3	17.7
	70	15.1	17.2	19.8	25	25	15.1	17.4	20.3
	75	14.8	16.9	19.5		30	14.9	17.0	19.8
15	15	16.7	19.3	22.8		35	14.7	16.7	19.4
	20	16.4	18.9	22.3		40	14.5	16.5	19.2
	25	16.2	18.6	21.9		45	14.3	16.3	18.9
	30	16.0	18.3	21.6		50	14.2	16.1	18.7
	35	15.9	18.1	21.3		55	14.0	15.9	18.4
	40	15.7	17.9	21.1		60	13.8	15.6	18.0
	45	15.6	17.8	20.9		65	13.6	15.3	17.6
	50	15.4	17.6	20.7		70	13.3	15.0	17.2
	55	15.3	17.4	20.4		75	12.9	14.6	16.7
	60	15.2	17.2	20.1	30	30	14.5	16.6	19.3
	65	15.0	16.9	19.8		35	14.2	16.2	18.8
	70	14.7	16.6	19.4		40	14.0	15.9	18.4
	75	14.4	16.3	18.9		45	13.8	15.6	18.1

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.	Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
30	50	13.6	15.4	17.8	45	65	11.4	12.5	14.1
	55	13.4	15.1	17.4		70	11.0	12.0	13.5
	60	13.2	14.8	17.0		75	10.6	11.6	13.1
	65	12.9	14.5	16.6	50	50	12.1	13.3	15.0
	70	12.6	14.1	16.1		55	11.7	12.9	14.5
	75	12.2	13.7	15.6		60	11.3	12.4	13.9
35	35	13.8	15.8	18.3		65	10.9	12.0	13.2
	40	13.5	15.4	17.8		70	10.5	11.5	12.6
	45	13.3	15.1	17.4		75	10.1	11.0	12.3
	50	13.1	14.8	17.1	55	55	11.3	12.4	13.6
	55	12.9	14.5	16.7		60	10.9	11.9	13.0
	60	12.7	14.2	16.3		65	10.5	11.5	12.4
	65	12.4	13.8	15.8		70	10.0	10.8	11.8
	70	12.0	13.4	15.3		75	9.5	10.3	11.3
	75	11.6	13.0	14.8	60	60	10.5	11.2	12.2
40	40	13.3	15.0	17.3		65	10.0	10.6	11.5
	45	13.0	14.6	16.8		70	9.5	10.1	10.9
	50	12.7	14.2	16.3		75	9.0	9.5	10.3
	55	12.4	13.9	15.9	65	65	9.4	10.0	10.7
	60	12.1	13.5	15.4		70	8.9	9.4	10.0
	65	11.8	13.1	14.9		75	8.3	8.7	9.3
	70	11.4	12.7	14.5	70	70	8.2	8.6	9.2
	75	11.0	12.3	14.0		75	7.6	7.9	8.4
45	45	12.8	14.2	16.2		75	6.9	7.2	7.6
	50	12.5	13.8	15.7					
	55	12.1	13.4	15.2					
	60	11.7	12.9	14.7					

TABLE

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TABLE XII.

For finding the Value of an Annuity upon One or more Lives, at the Rates of $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{1}{2}$, and $4\frac{3}{4}$, per Cent. Interest; supposing the Value, at the Rate of 4 per Cent. to be known from the preceding Tables.

Val. at 4 p. Cent.	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	$4\frac{1}{4}$	$4\frac{1}{2}$	$4\frac{3}{4}$
	add	add	add	sub.	sub.	sub.
6	0.3	0.2	0.1	0.1	0.2	0.3
7	0.4	0.2	0.1	0.1	0.2	0.3
8	0.5	0.3	0.1	0.1	0.3	0.4
9	0.7	0.4	0.2	0.2	0.4	0.6
10	0.9	0.6	0.3	0.3	0.5	0.7
11	1.1	0.7	0.3	0.3	0.6	0.9
12	1.3	0.8	0.4	0.4	0.7	1.0
13	1.4	0.9	0.4	0.4	0.8	1.2
14	1.6	1.0	0.5	0.5	0.9	1.3
15	1.7	1.1	0.5	0.5	1.0	1.4
16	1.9	1.3	0.6	0.6	1.1	1.6
17	2.1	1.4	0.7	0.6	1.2	1.8
18	2.3	1.5	0.7	0.7	1.3	1.9
19	2.5	1.6	0.8	0.7	1.4	2.0
20	2.7	1.7	0.8	0.8	1.5	2.2

PROBLEM

PROBLEM I.

To find the present Value of any Sum of Moncy, to be received at the End of a given Term of Years; discounting at the rate of 3, 4, or 5 per Cent. compound Interest.

SOLUTION.

Find, by *Tab. pag. 112*, the present value of 1*l.* to be received at the end of the given term; which multiply by the number of pounds proposed (cutting off 4 figures from the product, on account of the Decimals) then the result will be the value sought.

EXAMPLE. Let the sum proposed be 800*l.* the given term 7 years, and the rate of interest, 4 per Cent. Then the answer will appear to be .7599 multiplied by 800, or 607.9200*l.* that is, 607*l.*: 18*s.*: 5*d.*, nearly.

PROBLEM II.

To find the present Value of an Annuity, certain, for a given Number of Years; according to any of the Rates of Interest specified in the preceding Problem.

SOLUTION.

Seek, in *Tab. p. 114*. the number of year's-purchase answering to the given term of years; which, multiplied by the proposed annuity, gives the answer.

EXAMPLE. Suppose the annuity to be 100*l.* the number of years 7, and the rate of interest 4 per Cent. Then the value sought will be 6.002 multiplied by 100; or 600*l.*: 4*s.*

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PROBLEM III.

To find (according to Observations on the Bills of Mortality of the City of London) the Probability, or Proportion of Chance, that a Person, of a given Age, continues in Being a given Number of Years.

SOLUTION.

Let the given age be 40, and the number of years proposed 15.

Look, in *Tab. VI.* against 40 years, and also against 55 years, the age to which the person must arrive, if he live to the end of the given term; corresponding to which you will find the numbers 229 and 129, respectively; shewing that, of 229 persons who attain to the age of forty, only 129 reach the age of fifty-five: Now the excess of 229 above 129 being 100, it is evident that the odds, or the ratio of the chances, for and against, surviving the proposed term, will be as 129 to 100, or as 9 to 7 nearly: and, in the same manner, the answer will be found in any other case.

Note. Though this, and the following problem, are not immediately concerned about the business of Annuities, yet they are the foundation whereon the *whole* is grounded; and therefore do not improperly fill up the place here allotted them.

PROBLEM

PROBLEM IV.

To find (according to the aforesaid Observations) the Number of Years of Life, which a Person, of a given Age, may, upon an Equality of Chance, expect to enjoy.

SOLUTION.

Seek the given age, in Tab. VII. and against it you will have the answer, in years and decimal-parts.

Thus it will appear that, a person, 30 years old, may, upon an equality of Chance, expect 23.6 years more, for his share of life. *

PROBLEM

* By the expectation, or share, of life, is not here to be understood, that particular period, which a person hath an equal chance of surviving; this last being a different and more simple consideration. The expectation of a life (to put it in the most familiar light) may be taken as the number of years at which the purchase of an annuity, granted thereon, without discount of money, ought to be valued. Which number of years will differ more or less from the period above mentioned, according to the different degrees of mortality to which the several stages of life are incident.—Thus, it is much more than an equal chance (according to the table, of the probability of the duration of life, p. 2) that an infant, just come into the world, arrives not to the age of 10 years; yet the expectation, or share of life, due to it, upon an average, is near 20 years. The reason of which wide difference, is, the great excess of the probability of mortality in the first tender years of life, above that respecting the more mature and stronger ages — If the numbers that die at every age were to be the same, the two quantities above specified would also be equal; but when the said numbers become continually less and less, the expectation must, consequently, be the greater of the two.

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PROBLEM V. *

To find the Value of an Annuity for an assigned Life.

SOLUTION.

This Problem is resolved from *Tab. I.* by looking against the given age, under the proposed rate of interest; corresponding to which you will have the number of years-purchase required.

Ex. Let the given age be thirty-six, and the rate of interest 4 *per Cent.* and let the proposed annuity be 250*l.* Then the value thereof will appear to be 12.1 years-purchase, or 12.1 times 250*l.* Therefore, multiplying 250 by 12.1 (and cutting off one figure, upon account of the decimal) the answer comes out 3025*l.*

After the same manner the answer will be found in any other case, falling within the limits of the table. But, as there may be occasion, sometimes, to know the values of lives, computed at higher rates of interest than any there exhibited, the two following, practical, rules are subjoined; by which the problem is resolved, independent of tables.

RULE I. †

If the given age is not less than 45 years (nor greater than 85), subtract it from 92; multiply the remainder by the *perpetuity*, ‡ and divide the product by the said remainder added to 2½ times the *perpetuity*; then the quotient will be the number of years purchase required.

EXAMPLE.

* See Prob. 8. pag. 48.

† See Rule 1. pag. 117.

‡ It may, perhaps, be of use to some of my readers, to be informed here, that, by the *Perpetuity of an estate, or Annuity*, is understood the number of years-purchase of the *Fee Simple*; found by dividing 100 by the rate per Cent. at which interest is reckoned.

EXAMPLE.

Let the given age be 50 years, and the rate of interest 10 per Cent. Then, subtracting 50 from 92, there remains 42; which, multiplied by 10, the perpetuity, gives 420; this divided by 67, the remainder increased by $2\frac{1}{2}$ times (10) the perpetuity, quotes 6.3 nearly: Therefore, supposing the annuity to be 100*l*. its value, in present money, will be 630*l*.

RULE II. *

If the given Age is less than 45 Years (but not less than 10) take $\frac{9}{10}$ of what it wants of 45; which divide by the Rate per Cent. increased by 1.2; then, if the Quotient be added to the Value of a Life of 45 Years, found by the preceding Rule, you will have the required Number of Years-Purchase in this Case.

EXAMPLE.

Let the proposed age be 20 years, and the rate of interest 5 per Cent. Here, taking 20 from 45, there remains 25; $\frac{9}{10}$ whereof is 22.5; which, divided by 6.2, quotes 3.2; and this added to 9.8, the value of a life of 45 (found by Rule 1.) gives 13.0, for the number of years-purchase that a life of twenty ought to be valued at.

It will be needless, I presume, to offer any thing farther by way of example to the preceding rules; which bring out the conclusions so near the true values, computed from *real* observations, as seldom to differ therefrom by more than about $\frac{1}{10}$ or $\frac{1}{20}$ of a year's-purchase.

The observations here understood are those whereon the foregoing tables are grounded (as the most proper foundation for this *Place*). But, if any person is desirous of seeing a similar method of solution accommodated to the *Breslau* Observations (published by Dr. Halley, in N^o. 196 of the Philosophical Transactions, which

* See Rule 2. pag. 119.

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which are considerably different from those above-mentioned, deduced from the Bills of Mortality of the City of London) what follows may, perhaps, answer his expectation. The rule is thus, *Multiply the difference between the given age and 85 years by the perpetuity, and divide the product by $\frac{8}{10}$ of the said difference increased by twice the perpetuity; then the quotient will be the answer.* Which, from the age of eight, to eighty, will for the general part, come within less than $\frac{1}{8}$ of a year's purchase of the truth.

PROBLEM VI.

To determine the Value of an Annuity, granted for a given Term of Years, upon the Contingency of its ceasing on the Failing of a proposed Life, if this should happen before the Expiration of the said given Term.

SOLUTION.

Find, in *Tab. VIII. Column 1.* the age of the assigned life (or that nearest it) and find, in *Column 2,* the proposed term of years; against which last you will have the answer.

EXAMPLE I.

Let the age be 10, the number of years 15, and the rate of interest 5 per Cent. Then it appears, at one view, that the value sought will be 9.4 years purchase.

EXAMPLE II.

Suppose the given age to be 5 years, the proposed term 16 years, and interest as above: In which case it appearing that the value of a term of 15 years is 9.1, and that of a term of 20 years, 10.7, it is evident that the true answer here must be about 9.5.—But, that there may be no difficulty in allowing for the odd years (which is harder to do, as the differences are

are unequal) the table at pag. 6 is annexed, as a supplement to that preceding it: to comprehend the use of which the difference of the values, answering to the two nearest tabular numbers to the given term (in Tab. VIII) must be taken; which, in this case, is 1.6: and then, by entering Tab. IX. Column 1. with this difference, you will find against it, under the excess (1) of the given term above the next inferior tabular number, the value 0.4 to be added to that answering to the said inferior number in order to have the true conclusion.—After the same manner the value corresponding to the same age, and a term of 19 years will be found 10.4 But it may be proper to observe, that, to avoid trouble, it will be sufficient, in most cases, when the age given cannot be exactly found in the table, to take the tabular number that comes nearest to it, whether greater or lesser.

PROBLEM VII. *

To find the Value of an Annuity, for two assigned joint Lives, that is, for as long as they both continue in Being together.

SOLUTION.

Seek, in Tab. X. Column 1. the age of the youngest life (or that nearest to it) and find, in Column 2, the age of the elder; against which last you will have the number of years-purchase required.

EXAMPLE. I.

Let the two ages be 20, and 35, years, and the rate of interest 4 per Cent. Then it appears, at one view, that the value sought will be 9.8.

EXAMPLE

* See Prob. 9, pag. 48.

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EXAMPLE II.

Let the proposed ages be 25, and 37, years, and interest as before.

Here, if the age of the elder was to be $\left\{ \begin{smallmatrix} 35 \\ 40 \end{smallmatrix} \right\}$ the value sought would be $\left\{ \begin{smallmatrix} 9.6 \\ 9.1 \end{smallmatrix} \right\}$

Therefore, when the age is 37, it must be 9.4.

EXAMPLE III.

Suppose the given ages to be 32 and 57, and the interest of money at 3 per Cent. In this case, the value corresponding to the ages $\left\{ \begin{smallmatrix} 30 \\ 35 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} 57 \\ 57 \end{smallmatrix} \right\}$, appears to be $\left\{ \begin{smallmatrix} 7.6 \\ 7.4 \end{smallmatrix} \right\}$ Whence that answering to the given ages 32 and 57, must be about 7.5.

But, in order to avoid trouble, you may, upon occasion, add a year, or two, to one of the proposed ages, and subtract as much from the other, when they are nearly equal: but, if one of them much exceeds the other, it will then be sufficient to take the nearest number in the table for the lesser.

PROBLEM VIII. *

To find the Value of an Annuity, for the longest of two Lives, that is, for as long as either of them continues in Being.

SOLUTION.

Find, in Tab. XI. Column 1, the age of the youngest life (or that nearest it) and in Column 2, find the age of

* See Prob. 10. pag. 51.

Note. The Solutions to the several Problems inserted in the course of this work are, not improperly, divided into two different

of the elder; against which last you will have the number of years-purchase required.

EXAMPLE I.

Let the given ages be 15, and 40, years; and let the rate of interest be 4 per Cent. Then it is apparent that the value of an annuity upon the longest of two such lives will be 17.9, or nearly 18, years-purchase.

EXAMPLE II.

Let the ages propounded be 25 and 62, and the rate of interest as before: in which case we have 15.5 for the value of the annuity.

PROBLEM IX. (*)

To find the Value of an Annuity for three joint Lives, A, B, and C.

SOLUTION.

† Let A be the youngest, and C the oldest, of the three proposed lives: take the value of the two joint lives B and C (by Tab. X.) and find the age of a single life D, of the same value (by Tab. I.) then find the value of the joint lives A and D; which will be the answer.

EXAMPLE.

different classes: the first whereof, hereafter distinguished thus *, are general, and strictly true, according to any table of observations, or probability of life whatsoever: but the second sort, marked thus †, are near approximations only, yet such as are applicable, likewise, to any table of observation.

(*) See Prob. 11. pag. 54.

D

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EXAMPLE I.

Let the three given ages be 20, 35 and 44 years; and let the rate of interest be 3 *per Cent.* Then, the value of the two oldest joint lives B and C, will be found 9.0; answering to a single life (D) of 61 years: and the value of the joint lives A and D, which is 7.3 years-purchase, will be the value sought.

EXAMPLE II.

Suppose the three ages to be 17, 23 and 38, and the rate of interest 5 *per Cent.* Here the value of the joint lives B and C, will be 8.5; agreeing with that of a single life (D) of 55 years; whence the value sought, or that of the joint lives A and D, is in this case 7.0, or, just 7 years-purchase.

PROBLEM X. (*)

To find the Value of an Annuity for the longest of three Lives, A, B, and C.

SOLUTION.

† Let A be the youngest, and C the oldest, of the three proposed lives. Find the value of the joint lives B and C (*by Tab. X.*) and find (*by Tab. I.*) the age of a single life, D, of the same value: moreover find (*by Tab. XI.*) the value of the longest of the lives A and B, also that of the longest of the lives A and C, and likewise that of the longest of the lives A and D; then the last of these three values, subtracted from the sum of the two former, leaves the value sought.

EXAMPLE I.

Let the three ages be 20, 40 and 66 years, and let the rate of interest be 4 *per Cent.* Then the value
of

(*) See *Prob. 12. pag. 57.*

of the joint lives B and C will be found 5.8; answering to a single life (D) of 73 years: moreover the value of the longest of the lives A and B will be 17.2; that of the longest of the lives A and C 16.0; and that of the longest of the lives A and D 15.5. Therefore the value sought is 17.7 years purchase.

EXAMPLE II.

Let the given ages be 15, 28, and 37 years, and the rate of interest as before: Here (proceeding as in *Prob. 7. Ex. 3.*) the value of the two oldest joint lives will be had 9.2, equal to that of a single life (D) of $55\frac{1}{2}$ years (*by Tab. I.*) Hence, the three values specified in the latter part of the rule will, in this case, be 18.4, 18.0, and 17.4, respectively, and consequently the value sought, just, 19 years purchase.

PROBLEM XI. (*)

To find the Value of an Annuity granted upon three Lives, A, B, C, on Condition of its ceasing as soon as any two of them become extinct.

SOLUTION.

† Find (*by Tab. X.*) the value of each pair of joint lives, viz. of A and B, of A and C, and of B and C; then, from the sum of those three values, let twice the value of the three joint lives A, B, and C, (*found by Prob. IX.*) be deducted, and the residue will be the answer.

EXAMPLE

(*) See *Prob. 3. pag. 25.*

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EXAMPLE.

Suppose the given ages of A, B and C to be 20, 35 and 44 years, respectively; and let the rate of interest be 3 per Cent.

Here the value of the *joint lives*

$\left\{ \begin{array}{l} \text{A and B} \\ \text{A and C} \\ \text{B and C} \end{array} \right\}$ will be $\left\{ \begin{array}{l} 10.9 \\ 9.6 \\ 9.0 \end{array} \right\}$. The sum of which

three numbers is 29.5; moreover the value of the three *joint lives*, A, B and C is, in this case, 7.3. (See Prob. 9. Ex. 1.) Therefore 14.9 is the value sought.

PROBLEM XII.

The Value of an Annuity upon one, two, or three Lives, at the Rate of 4 per Cent. being known (from the preceding Problems); to find the Value of the same Life, or Lives, at the Rate of $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{1}{2}$, or $4\frac{3}{4}$, per Cent.

SOLUTION.

† This problem is solved by means of Tab. XII; from whence the value that ought to be added to, or subtracted from, the given value, at the rate of 4 per Cent. is had by bare inspection.

Thus it will appear that an annuity, upon one, or more lives, which at the rate of 4 per Cent. is worth 16 years purchase, will, at the rate of $3\frac{1}{2}$ per Cent. be worth 17 years purchase and three tenths. This method, though exact only, when applied to a single life, is sufficiently near in the case of two, or more, lives.

REMARK

REMARK.

As it is customary for an *Annuitant* to receive his money half-yearly in equal portions (instead of yearly) he hath, in this, a double advantage; for, besides the use of each first-half-yearly payment for 6 months, he also hath a chance of receiving one half-years value more, than if he was to be paid yearly. Now the value of both these considerations put together (let the rate of interest and the number of lives on which the annuity depends be what they will) will always amount to $\frac{1}{4}$ of a year's purchase. Therefore $\frac{1}{4}$ of a year's purchase must be always added to the values found by the foregoing problems, in order to have the true answer, when the payments are made half-yearly. And, if the payments are to be made quarterly, then $\frac{1}{8}$ of a year's purchase ought to be added to the value found by the preceding calculations.

OF

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OF REVERSIONS.

PROBLEM XIII.

To find the Value of the Reversion of an assigned Life after a given Term of Years.

SOLUTION.

* From the value of the proposed life, subtract the value of an annuity for the given term of years, on the contingency of its ceasing upon the extinction of the aforesaid life; the remainder will be the answer.

EXAMPLE.

A, aged 15 years, expects to enter upon an estate of 500*l.* per Annum, after the expiration of 10 years; which he is to hold thenceforward, during life; what is the present value of his expectation, reckoning interest of money at 4 per Cent.

Here the answer will be 8.3 years purchase, or 415*l.* For the value of the whole life A being 15.8 (by Tab. I.) and that of the first 10 years 7.5 (by Tab. VIII.) the difference will be 8.3, as above.

PROBLEM XIV.

To find the Value of the Reversion of an Annuity, for the Remainder of a given Term of Years, after an assigned Life.

SOLUTION.

* From the value of an annuity certain for the given term, subtract the value of the annuity for the said term, on the contingency of its ceasing upon the failing of the proposed life; the remainder will be the value of the reversion.

EXAMPLE

EXAMPLE.

A, aged twenty-five, who has the right of an annuity for 31 years certain, makes over the reversion thereof to B and his heirs, to enjoy the same after his decease, for the remainder of the said term. Now, in order to find the value of B's expectation, the value of an annuity *certain*, for 31 years, is to be found; which, at the rate of 4 *per Cent.* will be 17.58 (*by Tab. pag. 114.*) Moreover the value of an annuity for the same term, on the contingency of its failing on the extinction of A, will appear to be 12.9 (*by Tab. VIII.*) Therefore the value sought, in this case, is 4.68 years purchase.

PROBLEM XV. (*)

To find the Value of the Reversion of one Life after another.

SOLUTION.

* From the value of the life in expectation subtract the value of the two *joint lives*; the remainder will be the required value of the reversion.

EXAMPLE.

Let the age of the life in possession be 55 years, that of the life in expectation 20 years, the rate of interest 5 *per Cent.* and the proposed annuity 100*l.* Then, *by Tab. X.* the value of the two *joint lives* will be 6.9; which, subtracted from 13.0, the value of the life in expectation (*found by Tab. I.*) leaves 6.1 years purchase, for the value of the reversion. Which, multiplied by the proposed annuity, gives 610*l.* its value in present money.

Note. In the resolution of the preceding problem, the value that ought to be paid for putting in, or joining, a new life to one already in possession, is likewise determined

(*) See Prob. 13. pag. 65.

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determined. Thus, for example, if A and his heirs hold an annuity of 100*l.* upon a single life, of fifty-five, and they would put in another life, of twenty, to hold the annuity as long as either of the two lives continue in being; then the sum which ought to be paid, as an equivalent for that consideration, is 610*l.* the very value above exhibited.

PROBLEM XVI. (*)

To find the Value of the Reversion of two Lives after one.

SOLUTION.

* From the value of the three lives subtract the value of the life in possession, the remainder will be the value of the reversion.

EXAMPLE.

Let the age of the life in possession be 40 years, and the ages of the two lives in expectation 20 and 66 years; and let interest be supposed at 4 *per Cent.* In which case, the value of the three lives being 17.7 (*by Prob. 10, Ex. 1.*) and that of the life in possession 11.5 (*by Tab. 1.*) the answer comes out 6.2 years purchase: so that, if the annuity was to be 500*l.* the value of the reversion would be 3100*l.* Which sum also expresses the consideration that ought to be allowed for putting in two lives, of 20 and 66 years, to a life of forty, already in possession.

PROBLEM

(*) See *Prob. 14. pag. 66.*

PROBLEM XVII. (§)

To find the Value of the Reversion of one Life after two joint Lives.

SOLUTION.

* From the value of the life in expectation, subtract the value of the three *joint lives*, there will remain the value of the reversion.

EXAMPLE.

Suppose the age of the life in expectation to be 20 years, and the ages of the two lives in possession 35 and 44 years; and suppose interest of money to be at 3 per Cent.

Here the value of the three *joint lives* will be found 7.3 (by Prob. 9, Ex. 1.) which being deducted from 17.2, there rests 9.9 years purchase, for the value of the reversion.

PROBLEM XVIII. (§)

To find the Value of the Reversion of one Life after two.

SOLUTION.

* From the value of the three lives, subtract the value of the two lives in possession, there will remain the value of the reversion.

EXAMPLE.

Let 40 and 66 be the ages of the two lives in possession, and 20 that of the life in expectation, and let the rate of interest be 4 per Cent.

Then, (by Prob. 10, Ex. 1.) the value of the three lives will be found 17.7; from which taking 13.0, the value of the two lives in possession (found by Tab. XI.) the

(†) See Prob. 16. pag. 68.

(§) See Prob. 15. pag. 67.

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the remainder, 4.7, will be the number of years purchase at which the value sought is to be estimated. Which, therefore, also expresses the value that ought to be paid for putting in, or joining, a life of 20 years to two others of 40 and 66 years.

PROBLEM XIX. (§)

To find the Value of the Reversion of two joint Lives after one.

SOLUTION.

* From the value of the two joint lives in expectation, subtract the value of the three joint lives, there will rest the value of the reversion.

EXAMPLE.

Let the ages of the two lives in expectation be 20 and 44 years, and that of the life in possession 35 years; and let the rate of interest be 3 per Cent. Then, the value of the three joint lives being 7.3 (by Prob. 9, Ex. 1.) and that of the two joint lives in expectation 9.6 (by Tab. X.) the answer in this case will be 2.3 years purchase.

PROBLEM XX.

Q and his Heirs, as soon as any two of three given Lives A, B, C, become extinct, are to enter upon an Annuity, in order to enjoy the same, during the Life of the Survivor: to determine the present Value of the Expectation.

SOLUTION.

* To the sum of the values of all the single lives, add three times the value of the three joint lives; and from this sum deduct twice the sum of the values of each

(§) See Prob. 17. pag. 68.

each pair of joint lives (*viz.* of A and B, A and C, and B and C) the remainder will be the answer.

EXAMPLE.

Let the three given ages be 20, 35, and 44 years, and the rate of interest 3 *per Cent.* Then the values of the three *single lives*, by *Tab. I.* will appear to be 17.2, 14.1, and 12.5. And, by *Prob. IX. Ex. 1*, the value of the three *joint lives* is found 7.3: the treble of which value, added to the sum of the three former, gives 65.7. But the values of the three different pairs of *joint lives* are found, by *Tab. X.* to be 10.9, 9.6, and 9.0: the double of all which, taken from 65.7, gives 6.7 for the number of years purchase required.

PROBLEM XXI. (†)

A and B enjoy an Annuity, equally, betwixt them; which, after the Decease of either of them, is to belong entirely to the Survivor for Life; to find the Value of the Right of each in that Annuity.

SOLUTION.

* From the value of the life A or B, subtract half the value of the two joint lives; the remainder will be the value of the right of A, or B, accordingly.

EXAMPLE.

Let the age of A be 25 years, that of B 40 years, and the rate of interest 5 *per Cent.* Then the value of the two *joint lives* will be 8.2 (*by Tab. X.*) whereof the half is 4.1; this taken from 12.3, the value of the life A, leaves 8.2 for the expectation of A; but, being taken from 10.3, the remainder 6.2 will be the value of B's expectation.

PROBLEM

(†) See *Prob. 18. pag. 69.*

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PROBLEM XXII.

A given Annuity, after the Decease of A, is to be divided equally between B and C, during their joint Lives; and then is to go entirely to the last Survivor for Life; it is proposed to find the Value of B's Expectation.

SOLUTION.

* From the value of the reversion of the life B after the life A (*found by Prob. 15.*) subtract half the value of the reversion of the joint lives B and C after the life A (*found by Prob. 19.*) the remainder will be the true value of B's expectation.

EXAMPLE.

Let the given ages of A, B, and C be 35, 20, and 44 years, respectively; and let the rate of interest be 3 per Cent. Then the value of the life B after the life A will appear to be 6.3; moreover the value of the joint lives B and C after the life A will come out 2.3 (*Vid. Ex. to Prob. 19.*) Therefore the answer, in this case, is 5.15 years-purchase.

PROBLEM XXIII. (†)

A, B, and C share an Annuity equally amongst them; which, upon the Decease of any one of the three, is to be divided equally between the two Survivors during their joint Continuance, and then is to go intirely to the last Survivor for Life: to find the Value of the Right of A in the said Annuity.

SOLUTION.

* From the value of the life A, take half the sum of the values of the joint lives A B, and the joint lives A C; then to the remainder add $\frac{1}{3}$ of the value of the three

(†) See Prob. 20. pag. 73.

three joint lives A B C, and the sum resulting will be the true answer.

EXAMPLE.

Suppose the three ages to be 17, 23, and 38 years, respectively, and the rate of interest 5 per Cent.

Here the value of the life A will be 13.5; also that of the joint lives A and B will be 10.1 (by Prob. 7.) that of the joint lives A and C, 8.6, and that of the three joint lives A, B and C, 7.0 (by Prob. 9. Ex. 2.) Therefore the answer comes out 6.5 years-purchase.

Of successive Lives, and the Renewing of Leases.

PROBLEM XXIV. (†)

Supposing A to enjoy an Annuity for Life, and, at his Decease to have the nomination of a Successor, who is likewise to enjoy the Annuity for his Life; it is proposed to find the present Value of the two successive Lives.

* Multiply the value of the life A by the value of the life put in at his decease, and divide the product by the perpetuity; then let the quotient be subtracted from the sum of the said values, and the remainder will be the answer.

EXAMPLE.

Suppose the value of the life A, at 4 per Cent. to be 12 years-purchase, and that he hath the liberty, at his decease, to name a life to succeed him of the greatest value possible; which value, according to Tab. I. is 16.4, answering to an age of 10 years. Therefore, in this case, the product of the two values will be 196.8; which divided by 25, the perpetuity, quotes 7.872; this, deducted from 28.4, the sum of the said values, leaves 20.528 years-purchase for the true value of the two successive lives.

PROBLEM

(†) See Prob. 25. pag. 87.

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PROBLEM XXV.

To find the present Value of any Number of Lives in Succession.

SOLUTION.

* Multiply the value of each life (considered independent of the rest, without regard to time, or order of succession) by the rate of interest, *per Cent.* dividing the product by 100; subtract each quotient from unity, and multiply all the remainders continually together, subtracting their product, also, from unity; then the last remainder, multiplied by the perpetuity, will be the true value of all the successive lives.

Otherwise, thus.

* Find, in *Tab. pag. 114*, the number of years answering to the value of each of the given lives; then find in the same Table, the value of an annuity certain for the whole of all those years added together; which value will be the answer.

EXAMPLE.

Suppose the number of lives to be three; whereof the values, at 4 *per Cent.* are 8, 10 and 15 years-purchase. These being multiplied, each, by 4, and the products divided by 100, we have 0.32, 0.4, and 0.6 respectively: which, subtracted severally, from unity, leave 0.68, 0.6, and 0.4; whereof the continual product is 0.1632; this being also taken from unity there rests 0.8368; which, multiplied by 25, the perpetuity, gives 20.92 for the true answer.

Otherwise, by the second Method.

It appears that the number of years corresponding, in *Tab. pag. 114*.

to $\left\{ \begin{array}{l} 8 \\ 10 \\ 15 \end{array} \right\}$ Years Purch. is about $\left\{ \begin{array}{l} 9.8 \\ 13.0 \\ 23.4 \end{array} \right\}$ whereof the sum is 46.2; against which we have 20.9, the same as before.

PROBLEM

PROBLEM XXVI.

A given Sum of Money, is to be received (as a Legacy), on the Decease of B, who is now of a given Age: what is the Value thereof in present Money?

SOLUTION.

* Subtract the value of the life B from the *perpetuity*; then it will be, as the *perpetuity* is to the remainder, so is the proposed sum, to its value in present money.

EXAMPLE.

Let the age of B be 32 years, the rate of interest 4 *per Cent.* and the given sum 500*l.* Then the value of the life B being 12.7, and the *perpetuity* 25.0, it will be, as $25.0 : 12.3 :: 500*l.* : 246*l.*$ the value sought.

PROBLEM XXVII.

*A given Sum is to be received on the Extinction of the first, second, or third, of three assigned Lives; * to find the present Value of the Expectation.*

SOLUTION.

* Find (by Prob. 9, 10, or 11) the value of an annuity granted upon the assigned lives, on condition of its ceasing on the failing of the first, second, or third of those lives, according to the case proposed; which value subtract from the *perpetuity*, and then proceed as in the last problem.

EXAMPLE.

Let the given sum be 1000*l.* the rate of interest 3 *per Cent.* and the ages of the three lives 20, 35 and 44 years.

Here

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Here the value of an annuity to cease upon failing of the $\left\{ \begin{array}{l} 1^{\text{st}} \\ 2^{\text{d}} \\ 3^{\text{d}} \end{array} \right\}$ Life will be $\left\{ \begin{array}{l} 7.3 \\ 14.9 \\ 21.3 \end{array} \right\}$ by Prob. $\left\{ \begin{array}{l} 9 \\ 11 \\ 10 \end{array} \right\}$; which being subtracted, severally, from the *perpetuity* 33.3, there rests 26.0, 18.4, and 12.0; whence the answer comes out 780*l.* 552*l.* or 360*l.* according to the three aforesaid cases, respectively.

PROBLEM XXVIII.

A purchases a Leasehold Estate, upon one, two, or three assigned Lives, for a given Sum, on Condition that his Heirs fill up the Lease, continually, whenever all the Lives become vacant, paying a stated Fine: the Question is, to find the present Value of the whole Purchase allowed for the Estate, with the Annuity, or Rack-rent, answering thereto; supposing the Lives put in at each Renewal to be of the same given Ages.

SOLUTION.

* Find (*by Prob. 5, 8, or 10.*) the value of the life, or lives, first proposed; find also (*by the same*) the value of the life, or lives, with which the lease is to be constantly renewed; subtract the former of these from the *perpetuity*; then it will be as the latter is to the remainder, so is the fine proposed to the present value of all the fines: which added to the sum, paid at entering, gives the value allowed for the whole purchase. This being multiplied by the rate of interest, the product, divided by 100, will be the annuity, or rack-rent corresponding.

EXAMPLE.

Let the number of lives be three; whereof the ages are 15, 28, and 37 years; let the given sum be 2000*l.* the fine 1000, and the rate of interest 4 *per Cent.* Then the value of the three lives will appear to be 19.0
(*Vid.*

(*Vid. Ex. 2, Prob. 10*), which, subtracted from 25.0, the perpetuity, leaves 6.0 : therefore, if the value of the lives (found in like manner) with which the lease is to be renewed, be taken at 17.7 years purchase, we shall have $17.7 : 6 :: 1000l. : 339l.$ Hence the whole value allowed for the purchase is 2339*l.* and the annuity corresponding appears to be 93.56*l.* or 93*l.* 11*s.* 2*d.*

PROBLEM XXIX (†)

Supposing A to purchase an Estate, in Copyhold, upon any Number of assigned Lives, for a given Sum, on Condition that He and his Heirs fill up the Lease, continually, whenever a Life becomes vacant, by paying a proposed Fine : to find the present Value of the whole Purchase allowed for the Estate, with the Annuity, or Rack-rent corresponding ; supposing the Life put in, at each Renewal, to be of the same given Value.

SOLUTION.

* Subtract the sum of the values of all the single lives, upon which the lease is first granted, from the perpetuity multiplied by the number of those lives ; then it will be, as the given value of the life to be put in at each renewal, is to the remainder, found above, so is the fine proposed, to the present value of all the sums paid, for ever, for renewing ; which, added to the value paid at first entering, gives the total value of the purchase. This being multiplied by the rate of interest, the product, divided by 100, will give the annuity answering thereto.

EXAMPLE.

Let the number of lives be three, and their values (at 4 per Cent.) 10, 12, and 15 years-purchase ; also let the sum paid upon entering be 1600*l.* and the proposed

(†) See *Prob. 27. pag. 92.*

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posed fine 400*l.* and suppose the purchaser to have the liberty of renewing with lives of what ages he thinks proper, or most to his own advantage; which ages, according to our *Table I.* appear to be between 7 and 12 years; answering to 16.4 years-purchase. Therefore, the sum of the values of the three first lives being 37, and the perpetuity 25, we have here $16.4 : 38 :: 400 : 926.8$, or 926*l.* 16*s.* the present value of all the sums paid for renewing. Hence the whole value allowed for the purchase is 2526*l.* 16*s.* and the corresponding annuity 101*l.* 1*s.* 5*d.*

PROBLEM XXX.

The same being supposed as in the last Problem; to find how much the Rent-Roll of the first Proprietor's Estate ought to be encreased, on account of the Fines paid at Renewing.

SOLUTION.

* Multiply the present value of all the sums paid for renewing (*found as in the preceding problem*) by the rate per Cent. assigned; then the product, divided by 100, will be the answer. Which, in the case, there proposed, will be found 37*l.* 1*s.* 5*d.*

PROBLEM XXXI. (†)

There is an Estate, which, if A (who is a Minor) happens to die in a given Time, or before he attains to a certain given Age (suppose twenty-one) is, after his Decease, to go to B and his Heirs for ever; to find the Value of B's Expectation.

SOLUTION.

* From the perpetuity, subtract the value of a life of that age to which the expectation of B is limited; multiply the remainder by the present value of 1*l.* to be received at the end of the given term (*found by Tab. pag. 112.*) And let this last product be, again, multiplied

(†) See Prob. 23, pag. 82.

multiplied by the number of persons, in *Tab. VI.* arriving to the age above mentioned, dividing the product thence arising by the number, in the same table, answering to A's present age; then to the quotient add the value of the life A, and subtract the sum from the perpetuity; the remainder will be the true value required.

EXAMPLE.

Let the age of A be 8 years, the proposed estate 500*l.* per Annum, and interest at 5 per Cent. Here the value of a life of twenty-one (the age to which B's expectation is limited) being 12.9, and the perpetuity 20, the difference will be 7.1; which multiplied by 0.5303, the present value of 1*l.* to be received 13 years hence (when A arrives to the age of twenty-one) gives 3.76513, or 3.76, nearly: this being multiplied by 355, the product, divided by 422, gives 3.16; to this adding 14.3, the value of the life A, and deducting the sum from 20, the perpetuity, there results 2.54, or $2\frac{1}{2}$ years-purchase nearly, for the required value of B's expectation.

Of Reversions depending upon the Probability of one particular Life, in Possession, surviving the rest.

PROBLEM XXXII.

B, who is of a given Age, will, if he lives till the Decease of A, whose Age is also given, become possessed of an Estate of a given Value; to find the Worth of his Expectation in present Money.

SOLUTION.

† Find (by *Tab. X.*) the value of an annuity on two equal joint lives, whereof the common age is equal to the age of the older of the two proposed lives A and B; which value subtract from the perpetuity, and take half the remainder; then say, as the expectation of the duration of the younger of the two lives A and B, found against the age corresponding, in *Tab. VII.* is to that

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of the elder, so is the said half-remainder, to a fourth proportional; which will be the number of years-purchase required, when the life B in expectation is the older of the two. But, if B be the younger, then add the value so found to that of the *joint lives* A and B (*found by Tab. X.*) and let the sum be subtracted from the *perpetuity*, and you will also have the answer in this case.

EXAMPLE I.

Suppose the age of A to be thirty, that of B forty, the rate of interest 4 *per Cent.* and the given legacy 5000*l.* or 200*l.* *per Annum.* Then the value of two equal *joint lives* of forty being 8.1, and the *perpetuity* 25, the remainder, or difference, will here be 16.9; whereof the half is 8.45: therefore it will be, as $23.6 : 19.6 :: 8.45 : 7.02$, years-purchase, or 1404*l.* the required value of B's expectation.

EXAMPLE II.

Let the age of A be forty, and that of B thirty, and the rest as in the preceding example. Here the value of the *joint lives* A and B will be 8.8; which added to 7.02 (*found above*) the sum will be 15.82; whence the answer in this case, is 9.18 years-purchase, or 1836*l.*

PROBLEM XXXIII. (†)

C and his Heirs are entitled to an Estate of a given Value, upon the Decease of B, provided B survives A: to find the Value of their expectation in present Money.

SOLUTION.

† Find (*by Tab. XI.*) the value of an annuity upon the longest of two equal lives, whereof the common age

(†) See this problem solved in a different manner, in pag. 104.

age is that of the older of the lives A and B; which value subtract from the *perpetuity*, and take half the remainder; then it will be, as the expectation of duration of the younger of the lives A and B (*found by Tab. VII.*) is to that of the older, so is the said half-remainder, to the number of years-purchase required, when the life B is the older of the two: but if B be the younger, then, to the number thus found, add the value of an annuity on the longest of the lives A and B, and subtract the sum from the *perpetuity*, for the answer in this case.

EXAMPLE.

Let the age of A be thirty, that of B forty, the rate proposed 4 *per Cent.* and the given estate 200*l.* *per Annum.*

Here, the value of the longest of two lives, aged forty each, will be found 15.0; which taken from 25, leaves 10 for the remainder. Therefore, it will be $23.6 : 19.6 :: 5 : 4.15$, the number of years-purchase required; answering to 83*ol.*

But if A had been forty and B thirty, the value sought would have been 4.95×200 , or 990*l.* Because the value of the longest of the lives A and B is 15.9 years-purchase.

PROBLEM XXXIV.

C is to enter upon a given Annuity for life, on the Decease of B, in case B survives A: to determine the Value of his Expectation in present Money.

SOLUTION.

† Case 1^o If the Life C, in Expectation, be older than either of the other two: find, by Prob. 18, the value of the reversion of the life C after the longest of the lives A and B; one half of which will be the answer.

Case 2^o If the Life C be younger than one, or both, of the other: then find the reversion of the life C after the

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the longest of two equal lives of the same age with the oldest of A or B; which multiply by the expectation of duration of the said oldest life (*given in Tab. VII.*) and divide the product by twice that of the younger; so shall the quotient be the answer, *in case B is older than A*: but, if B be younger than A, let the said quotient be subtracted from the value of the reversion of the life C after the longest of the lives A and B, and the remainder will be the answer.

EXAMPLE I.

Let the age of A be 30; that of B, 40; and that of C, 50 years; and suppose the rate of interest to be 3 *per Cent.*

Then, *according to Case 1*, the value of the longest of the three lives will be found to be 19.6 (*by Prob. 10.*) From which deducting 18.4 the value of the longest of the two lives A and B, the remainder 1.2 will be the value of the reversion of the life C after A and B; the half of which, or $\frac{6}{10}$ of a years-purchase, is therefore the value required.

EXAMPLE II.

Suppose the ages of A, B, and C to be 66, 40, and 20, respectively, and the rate of interest 4 *per Cent.*

Here, *according to Case 3*, we are first to find the value of the reversion of a life of twenty after the longest of two equal lives of 66; which (*by Prob. 18.*) will come out 7.0: this being multiplied by 10.1, and the product divided by 39.2 (*vid. Tab. VII.*) there results 1.8 nearly. Which, as B is here younger than A, must be subtracted from (4.7) the value of the reversion of the life C after the longest of the lives A and B (*found by Prob. 18.*) whence we get 2.9 for the true answer in this case.

PROBLEM

PROBLEM XXXV.

C is to enter upon a given Annuity, for Life, on the Decease of B, provided the Latter is survived by A : to find the Value of his Expectation.

SOLUTION.

† Find, by the last Problem, the value of the reversion of the life C after the life B, on the contingency of B's surviving A ; which, subtract from the absolute value of the reversion of the life C after the life B (found by Prob. 15.) the remainder will be the answer.

EXAMPLE.

Let every thing be supposed as in the second example of the last problem : then the former of the (above mentioned) reversions will appear to be 2.9 : and the latter (by taking 9.2, the value of the joint lives B and C, from 14.8, that of the life C) will be found 5.6. Therefore 2.7 years-purchase is the value sought.

PROBLEM XXXVI. (†)

A and B are possessed of an Annuity between them : which, if B survives A, is afterwards to be divided equally between B and C, during their joint existence ; and then is to go intirely to the last Survivor for Life : to find the value of C's Expectation.

SOLUTION.

† Find, by Prob. 34, the value of the reversion of the life C after the life B, on the contingency of B's surviving A ; to which add half the value of the reversion of the joint lives B and C after the life A (found by Prob. 19.) the sum will be the value sought.

EXAMPLE

(†) See Prob. 29. (pag. 105.) which is solved in a different manner.

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EXAMPLE.

Suppose the given ages of A, B, and C, to be 66, 40, and 20 years, respectively, and the rate of interest 4 per Cent.

Then the value of the former of the two reversions above mentioned will be 2.9 years-purchase (*vid. Ex. 2, to Prob. 34*) and that of the latter (by taking 5.2, the value of the joint lives A, B, and C from 9.2, the value of the joint lives B and C) will come out 4.0: therefore 4.9 years-purchase is the required value of C's expectation.

PROBLEM XXXVII.

C engages to pay A a given Annuity, till such Time (if they both live so long) as the latter comes to the Possession of an estate, which he is intitled to on the Decease of B: in consideration whereof, A, on his Part, obliges himself, if he lives to enter upon the said Estate, to cause a given Annuity to be paid back to C for Life (but if either C, or A, happens to die before B, then the Money advanced by C is to be wholly lost to his Heirs) The Question is, to find the Advantage, or Disadvantage of either Party in such a Contract.

SOLUTION.

† Find, by *Prob. 9*, the number of years-purchase of the three joint lives A, B and C; which multiply by the annuity that A is to receive, the product will be the value of A's whole expectation from C.

Find also, by *Prob. 34*, the value of the reversion of the life C after the life B, on the contingency of B's surviving A; which subtract from the value of the reversion of the life C after the life B (*found by Prob. 15.*) then the remainder, multiplied by the annuity paid back to C, will give the whole of C's expectation from A: and the difference of the two values thus found will consequently be the answer.

EXAMPLE

EXAMPLE.

A young gentleman* (A) aged twenty-five, having greatly disobliged his father (B) a gentleman of sixty, to whom he is sole heir, is forced to contract with a certain person (C) aged 35, for an annuity of 200*l.* (in order to a support) which he engages to repay, after his father's decease, with another of 300*l.* to continue during the Life of C; according to the conditions specified in the Problem.

Now we are first to find the value of the three joint lives; which, at the rate of 4 *per Cent.* comes out 5.6 years-purchase; and this, multiplied by 200, gives 1120*l.* for the whole value of A's expectation.

Moreover, the value of the reversion of the life C after the life B, on the contingency of B's surviving A, will come out one year's-purchase, very nearly: And the value of the reversion of the life C after the life B (without restriction) will appear to be 5.8 years-purchase (*by Prob. 7*). Hence 4.8, multiplied by 300, which is 1440*l.* will be the total value of C's expectation: Who therefore gains 320*l.* by the contract.

PROBLEM XXXVIII.

C, if he lives till the Decease of B, is to receive a given Legacy, in case A is then extinct; to determine the Value of his Expectation in present Money.

SOLUTION.

† Case 1^o If the life C be the oldest of the three: From the value of an annuity on the life C, take the value of the two joint lives B and C; multiply the remainder by the given sum, or legacy, and divide the product by twice A's expectation of duration (*found in Tab. VII*); the result will be the value sought.

Case 2^o If the life B be the oldest of the three: Then, from the value of an annuity, for as many

G
years

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years of C's life as are expressed by the double of B's expectation of duration (*found by Tab. VII. and VIII.*) subtract the value of the two *joint lives* B and C; multiply the remainder by the given sum, or legacy, and divide the product by twice the expectation of A's duration, *as in the preceding case.*

Case 3^o *If the life A be the oldest of the three:* then find the value of the life C, if older than B; otherwise, find (*by Tab. VIII.*) the value of as many years thereof, as are expressed by the double of B's expectation of duration: And, from the value thus found, let the value of the *joint lives* A and C be subtracted; multiply the remainder by the given legacy, then the product, divided by twice B's expectation of duration, will be the answer in this case.

EXAMPLE I.

Let the age of C be 15 years, that of B 70, and that of A 45; and let the proposed legacy be 1000 *l.* and the rate of interest 3 *per Cent.*

This example, it is plain, belongs to case 2^o. according to which we must first find B's expectation of duration (*by Tab. VII.*) which appears to be 8.8, and the double of it 17.6: Now (*by Tab. VIII.*) the value of 17.6 years of C's life is found 11.9; and (*by Tab. X.*) the value of the joint lives B and C appears to be 6.0; which last value, subtracted from the former, leaves 5.9; this multiplied by 1000 (the given legacy) produces 5900; which being divided by 35.6, the double of A's expectation of duration, there results 165 *l.* for the required value of C's expectation.

EXAMPLE II.

Suppose the age of C to be 15 years, that of B 45, and that of A 70; suppose also the given legacy to be 1000 *l.* and the rate of interest 3 *per Cent.*

Here, according to case 3^o, we must first seek, in *Tab. VII.* for B's expectation of duration; this appears to

to be 17.8, and the double thereof 35.6: Now the value of 35.6 years of the life C (by Tab. VIII.) will be found 16.7; from which subtracting (6.0) the value of the joint lives A and C (found by Tab. X.) there remains 10.7; this, multiplied by 1000, gives 10700; which, divided by 35.6, quotes 300 l. for the required value of C's expectation, in present money.

PROBLEM XXXIX.

Q, in case he lives till R and S are both extinct, is to receive a Legacy of a given Value; to determine the Worth of his Expectation in present Money.

SOLUTION.

† Find, by the last problem, the expectation of Q, on the contingency of R's surviving S; and also his expectation, on the contingency of S's surviving R: The sum of which values will, consequently, be the whole of his expectation in the present case.

EXAMPLE.

Let the age of Q be 15 years, that of R 45 years, and that of S 70; also suppose the given legacy to be 1000 l. and the rate of interest 3 per Cent.

Here the first part of Q's expectation, depending on R's surviving S, appears, from Ex. 2. of the preceding Problem (by substituting Q for C, R for B, and S for A) to be 300 l.

And the latter part of his expectation, depending on S's surviving R, appears, from Ex. 1 (by substituting Q for C, S for B, and R for A) to be 165 l. Therefore the answer is 465 l.

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PROBLEM XL.

Q and R, if they live till the Decease of S, are to receive, each of Them a given Legacy; but, in Case one of Them dies before Him, then the Whole is to go to the Survivor: To determine the Value of Q's Expectation.

SOLUTION.

† Find, by *Prob. 32*, the value of Q's expectation on the Sum assigned to himself (depending on the chance of his surviving S, without further restriction): Then find, by *Prob. 38*, the remaining part of his expectation, on the sum assigned to R (depending on the contingency of S's surviving R:) The sum of the two values so found will consequently be the answer.

EXAMPLE.

Suppose Q to be 15 years of age, R 45, and S 70; suppose also the legacy of Q to be 600 *l.* That of R 1000 *l.* and the rate of interest 3 *per Cent.*

In the first place we are to find the present value of 600 *l.* (or an annuity of 18 *l.*) to be received (or entered upon) by Q, at the Decease of S: Which value comes out 23.37 years-purchase, or 420 *l.* Moreover, by *Ex. 1, to Prob. 38*, the latter part of Q's expectation, depending upon the chance of his receiving the 1000 *l.* assigned to R, will appear to be 165 *l.* So that the whole is 585 *l.*

PROBLEM

PROBLEM XLI.

Q, R, and S share an Annuity equally among them; which, upon the Decease of any one of the three, is to be divided equally between the two Survivors during their joint Continuance; and then is to go intirely to the last Survivor and his Heirs for ever: to determine the Value of the Right of Q in the said Annuity.

SOLUTION.

† First find, by *Prob. 39*, that part of Q's expectation which depends on his becoming possessed of the whole annuity, on being the last survivor: to which value add half the sum of the values of the joint lives QR, and QS; and from the aggregate let $\frac{2}{3}$ of the value of the three joint lives QRS be subtracted; the remainder will be the value sought.

EXAMPLE.

Suppose Q to be 15 years of age, R 45, and S 70; suppose moreover the annuity to be 300*l.* and the rate of interest 3 *per Cent.* according to which the value of the perpetuity is 10000*l.* Whence, by the example to the problem above quoted, the former part of Q's expectation will appear to be 4650*l.* We are next to find the values of the joint lives QR, and QS; these, by *Tab. X.* appear to be 9.6, and 6.0, years-purchase, respectively: whereof the half sum is 7.8 years-purchase, or 2340*l.* Moreover, by *Prob. 9*, the value of the three joint lives QRS, will be found 5.2 years-purchase, or 1560*l.* and $\frac{2}{3}$ thereof 1040*l.* And so, by adding together 4650 and 2340, and taking 1040 from the sum, the answer comes out 5950*l.*

SCHOLIUM

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SCHOLIUM.

*Wherem the Reasons of what is most material and difficult
in the preceding Solutions, are explained.*

The solutions to the several problems proposed in the foregoing pages being delivered in a practical manner, without their investigation (in order to render the work as useful as possible, and acceptable to those, to whom the sight of an algebraical process would, at best, afford no satisfactory, or pleasing idea) and there being, in the number of the said solutions, some few that even the mathematical reader (whom I should be loth to leave dissatisfied) may not readily see into the reasons of; I shall here, according to my promise, put down the invention of what seems most material, and necessary to be explained. It is not indeed my design to insist, in this place, on the first principles of the laws of chance, and the general methods of computation whereon the subject is grounded; but, to *such* as want information therein, shall take the liberty to recommend my *Doctrine of Annuities*, printed in 1742. Nevertheless, as the merit of a work of this nature, when applied to the common uses of life, consists in its conformity with truth and *real* observations, it is but reasonable that, for the general satisfaction of those into whose hands this Tract may happen to fall, something should be, first of all, said relative to the observations whereon the foregoing tables have their foundation.

The first of these tables (shewing the probability of the duration of life) on which the rest depend, was computed from 10 years observations on the Bills of Mortality of the City of *London*; and is the very same, in effect, with that printed in my Book of Annuities above mentioned; though adapted to a different radix. Both these tables agree with that first of all published by Mr. *Smart*, whom I have followed, except in very low ages; where it seemed necessary to make some alterations.

alterations. I am sensible, indeed, that these alterations have been condemned, as arbitrary and without foundation: but that some such allowance ought to be made, on account of the great and unequal afflux of people, of different ages, to town, is, I think, sufficiently evident from the following reasons.

First, it is to be observed, agreeable to what I have elsewhere advanced, that the said continual resort of strangers to town would no way influence the values of annuities deduced from observations on the Bills of Mortality, if those arriving and settling *there*, at the several different ages of life, were to be in the same proportion as the whole number of the living of the same ages; since it is not on the greatness of the numbers that die at each particular age, but on their ratio that the whole of the business depends.

To render this as obvious as may be, by a familiar example; suppose A and B to be two equal cities, with regard to the number of their inhabitants, and let half the inhabitants of the former who are above the age of twenty, be supposed to remove to the latter; then it is plain that the Bills of Mortality of B, for all ages above twenty, will be immediately increased by one half; and so the same proportion still preserved: for the equimultiples of quantities are to one another as the quantities themselves.

But, with respect to ages below twenty, the Bills still continuing the same, the numbers here cannot be compared with those above, in order to obtain the true ratio of the probabilities of mortality at the different ages of life; seeing these last are too little by one half.

It is evident from hence, that, if there be any part of life wherein the number of those that remove to town falls short of the proportion above specified, the Bills of Mortality, for that interval, will not truly exhibit the probability of mortality without some allowance or correction.

Now it is certain, from manifold experience, that very few persons come to live in town under the age
of

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of fifteen, in comparison of the numbers that arrive there, between the ages of fifteen and thirty; though the number of all the living comprehended in the former period is much greater than that in the latter.

Add to this, that the bills of mortality, with respect to small ages, are also lower than they would otherwise be, on account of a great number of youth, of the better sort, who are sent into the country for the benefit of air and education. These, at their return, together with the arrival of a multitude of working people (who, after having served an apprenticeship in the country, are willing to learn experience, and try their fortune in town), very much increase the body of inhabitants: And it is chiefly to this consideration that the great increase in the bills of mortality after the age of twenty, like a rivulet swollen by a sudden rain, is to be ascribed: For I dare affirm, that there is nothing in man's constitution whereby so great a disproportion during so short an interval, can be, physically, accounted for: Nor does it seem, in the least, reasonable to suppose that a person, of twenty-five years of age, has more than 4 times a greater chance to die within the compass of a year, than one of fifteen; and yet this is the case according to the table, I have been condemned for altering—As to the reason of my beginning the table with the number 1280 (instead of 1000) it was with no other intent than to avoid trouble in making, what I judged, the necessary alterations. Nor does it at all signify what number is made the radix of a table of this kind, provided the same proportions are preserved throughout. However, in conformity to custom, I have here been at the pains to reduce the said table to the common radix.

Amongst those who have been severe on the above alterations, there is a certain foreign gentleman, *Mr. Parcieux*, Member of the Royal Academy of Sciences at *Paris*, whom I must not pass by without a more particular notice. This gentleman, in a book intitled

intitl'd *Essai sur les Probabilités de la vie humaine*, (as appears by an account given of it in the *Memoirs* of the said Academy, 1746) accuses me of having assigned much too great a probability of mortality to the youngest ages; and further affirms that *this error* (for such he calls it) arises from omitting to include, in the account, those persons who, having escaped the weakness of infancy, go out of the bills of mortality and die elsewhere.

Whether, or no, the fact be as it is here represented, this writer does not appear to be a sufficient judge: he does not seem to be apprized, that there is not, perhaps, a city in the world so fatal to the infant state as the city of *London* (owing, too much, to the intemperance of parents and a profuse irregular manner of living). Besides, were the bills of mortality of a place to be ever so regular and well kept, it ought to be remembered that there can be no arriving at an absolute certainty in matters of this nature. Now, were it for these reasons alone, I could not help looking upon this gentleman's assertions (to view them in the most favourable light) as a little too peremptory; especially where he tells the world that, *I was not aware, that, by avoiding Sir W. Petty's fault, I myself fell into the contrary mistake*. I declare, I am not conscious of any such mistake; nay I can positively affirm, that the reason he assigns for it is altogether chimerical: Nor is there any thing I have said in my preface, or elsewhere, that can justify such a construction. But, what need have I to insist upon this! every body here knows that the number of those who are born in town and die elsewhere, is inconsiderable in comparison of the number of those who, on the contrary, leave the country and die in town: So that, had they been *actually* disregarded, no such error as he speaks of could possibly have arisen from thence.

The seventh table, in this work, *shewing the number of years that a person of any age, may, upon an equality of chance, expect to enjoy*, is deduced from the first; by a method of computation no way different from

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that, for finding the value of an annuity upon a single life, when money is supposed to bear no interest; only the value of one half-year extraordinary is to be added *here*; because an *Annuitant*, who receives his money yearly, has an equal chance of living half a year after the time of the last payment.

As to the manner of computing the tables for the values of single and joint lives; this I have shewn (with proper expedients to facilitate the trouble of such a work) in my Book of Annuities aforesaid, to which I must beg leave to refer.—The first eight problems, relating only to the use of these tables, need no other explanation than is already given them.

The 9th problem, *determining the value of three joint lives*, is the first that seems necessary to be taken notice of *here*: whereof the solution is not, indeed, strictly conformable to the table of observation, being only an approximation, but such an one as answers very near the truth; and which may be applied with advantage to almost any hypothesis or table of observations. The reasonableness of the method of proceeding is evident from the nature of the subject, without calling in the assistance of any kind of Computation. And in a number of examples respecting lives of different ages, I scarce ever found the error to exceed $\frac{1}{8}$ of an years-purchase.

The solution to our 10th problem, *for determining the value of the longest of three lives*, is equally exact, and built upon the same foundation with the preceding one. For, the value of the longest of three lives being truly and universally expressed by $A + B + C - (AB) - (AC) - (BC) + (ABC)$; (where (AB) , (AC) , and (BC) denote the values of each pair of *joint lives*, &c. *Vid. Dat. Ann. p. 23*); it follows that the said value, if (according to *Prob. 9*) a single life D be taken equal in value to the two *joint lives* (BC) , will also be expressed by $A + B + C - (AB) - (AC) - D + (AD)$
or

or by its equal $\overline{A + B - (AB)} + \overline{A + C - (AC)} - \overline{A + D - (AD)}$: but $\overline{A + B - (AB)}$ is the value of the longest of the two lives A and B, and $\overline{A + C - (AC)}$, that of the longest of the lives A and C, &c. Whence the whole is manifest.

The solution to the next problem, and those that follow afterwards on reversion and successive lives, are either such as are evident from plain reasoning, exclusive of mathematical principles, or such whose demonstration I have already given elsewhere: for which reasons it seems needless to say any thing further about them in this place.

But in those kind of reversion considered in the thirty-second and the following problems, depending upon the probability of one particular life, in possession, surviving the rest, the reader must, I apprehend, have met with some difficulty, the rules there laid down being derived by virtue of a certain hypothesis; which, however, answers very near the truth.

In this hypothesis the numbers of the living at each several age, differing by an equal interval of one year, are supposed to form an arithmetical progression (or a series whose terms decrease continually by a common difference). This assumption was first adopted by Mr. *De Moivre*, and applied by him to good purpose in calculating the values of lives according to the *Breslaw* observations (mentioned in p. 21), with which it, indeed, much better agrees than with those deduced from the Bills of Mortality of the City of London.

But, though the hypothesis aforesaid cannot be applied with equal advantage here, without some farther consideration, seeing the differences of the numbers in *Tab. VI*, are far from being the same throughout; yet, by taking the mean of the said differences, the value of a *real* life may be converted into that of an equal *imaginary* one, conformable to the said hypothesis.

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This is actually done in *Tab. VII.* in which you have the number of years that a person of any age may, upon an equality of chance, be expected to enjoy: the double whereof is the *complement* of an equal *imaginary* life, according to the said hypothesis; where, by the *complement*, is to be understood the number of years that *such* a life has a chance of continuing in being.

Thus it will appear, that the value of a *real* life of fifty, is equal to that of an *imaginary* one whose *complement* is 32 years. This indeed is exact only when money is supposed to bear no interest; in all other cases, the latter is only an approximation, but so near as to come within a sufficient degree of exactness, as will be seen from the annexed table, wherein are exhibited the values of annuities for every tenth year of life, both according to real observation, and to the said hypothesis.

Age.	Value at 5 per Cent. by Obs.	Value at 5 per Cent. by Hyp.	Value at 4 per Cent. by Obs.	Value at 4 per Cent. by Hyp.	Value at 3 per Cent. by Obs.	Value at 3 per Cent. by Hyp.
10	14.3	14.2	16.4	16.3	19.0	19.1
20	13.0	13.1	14.8	15.0	17.2	17.2
30	11.6	11.9	13.1	13.4	15.0	15.1
40	10.3	10.7	11.5	11.9	13.2	13.4
50	9.2	9.6	10.1	10.4	11.4	11.5
60	7.9	8.1	8.4	8.6	9.2	9.4
70	6.2	6.3	6.5	6.6	6.9	7.1

By inspecting of which table it will appear that the error seldom exceeds $\frac{1}{40}$ part of the whole value, and is the least of all, where the interest of money is lowest, agreeable to what was before intimated.

But before I proceed to the use of the foregoing hypothesis in the resolution of the several problems above

above mentioned, I shall put down the investigation of a known theorem, for finding the value of any proposed life, according to the said hypothesis: which (being done without the summation of series) may perhaps appear more obvious than that delivered, at p. 15, of my former Tract.

Let, therefore, a be taken to denote the complement of the proposed life A (as above defined) also let P denote the value of the perpetuity, and r the amount of 1*l.* in one year: and, in order to render the operation as easy as may be, suppose another person (Q) and his heirs to be intitled to the reversion of the annuity for ever after the demise of A : then, if the value of the said reversion be computed, and subtracted from the perpetuity, the remainder will evidently be the required value of the life A . But to find the true value of this reversion, we must determine the several parts thereof, which depend on the contingency of A 's becoming extinct in the first, second, third, and each succeeding year.

Thus, if the life A fails in the first year, whereof the probability (according to the hypothesis) is $\frac{1}{a}$, then, as Q and his heirs become possessed of the whole estate, or perpetuity (without any deduction) it is plain that the true value of their expectation, on the contingency of A 's becoming extinct the first year, will be $\frac{1}{a} \times P$.

Again, if the life A fails the second year, whereof the probability is also $\frac{1}{a}$, the value of Q 's expectation will be the value of the perpetuity discounted for one year: and so the expectation on this event is $\frac{1}{a} \times \frac{P}{r}$.

And in the same manner it will appear, that the expectation on the contingency of A 's dying in the 3^d, 4th, 5th, &c. year, will be represented by $\frac{1}{a} \times$

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$\frac{P}{r^2}$, $\frac{1}{a} \times \frac{P}{r^3}$, $\frac{1}{a} \times \frac{P}{r^4}$, &c. respectively; and consequently, that the sum of all these, or its equal $\frac{rP}{a}$ into $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \dots + \frac{1}{r^a}$, will be the total value of the reversion. But the series $\frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \dots + \frac{1}{r^a}$ is known to express the value of an annuity certain for a years: which let be denoted by M ; then the value of the reversion being $= \frac{rP}{a} \times M = \frac{P+1 \times M}{a}$ (because $P = \frac{1}{r-1}$) the value of the proposed life A , in possession, will consequently be $P - \frac{P+1 \times M}{a}$.

Q. E. I.

I come now to the investigation of the solutions of the thirty-second, and the succeeding problems.

In order to which let a and b be taken to represent the complements of two imaginary lives, of the same values with the two real ones A and B (*Vid. Prob. 32.*) Moreover, let S denote the given sum, or legacy, to be received on the decease of A , if B be then living; and let r represent the proposed rate, or 1% increased by its interest for one year.

Now, to obtain an answer to the said Problem, we must find the value of B 's expectation on each particular year, according to the probability he has of receiving the Legacy (S) at the end of that year: and each of these values must be again considered in two parts, according to the chance of both the lives failing, or not failing, within the compass of the same year.

The probability that both A and B are extinct at the end of the first year being $\frac{1}{a} \times \frac{1}{b}$, the expectation

thereon

thereon would be $\frac{1}{a} \times \frac{1}{b} \times \frac{S}{r}$ (or $\frac{S}{abr}$) was it a certainty that A would drop first; but as there is an equal chance for the contrary (under the above restrictions,) the true expectation can therefore be only $\frac{S}{2abr}$. In the same manner, the expectation, on the chance of both the lives failing in the 2d, 3d, 4th, &c. years, will be $\frac{S}{2abr^2}$, $\frac{S}{2abr^3}$, $\frac{S}{2abr^4}$ &c. And the

sum of all these $\frac{S}{2ab} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \text{\&c.}$ will consequently be the whole of B's expectation, on the contingency of both the lives dropping in one and the same year: where the series is to be continued to as many terms as are expressed by the *Complement* *b* of the oldest life.

But now, with regard to the other part of B's expectation, depending on the probability of his surviving the year wherein A expires (which is, by far, the most considerable), it is evident that the part thereof which depends upon the chance of his coming into possession at the end of any proposed year, is compounded of the probability of his being then alive, and that of A's dying within the compass of the same year (allowing also for the discompt of money to that time).

Hence the expectation on the proposed sum *S*, on the contingency of receiving it at the end of 1, 2, 3, 4, &c. years, will be expressed by $\frac{b-1}{b} \times \frac{1}{a} \times \frac{S}{r}$, $\frac{b-2}{b} \times \frac{1}{a} \times \frac{S}{r^2}$, $\frac{b-3}{b} \times \frac{1}{a} \times \frac{S}{r^3}$, &c. respectively.

the aggregate of all which values, added to the former part of B's expectation (found above) gives

$\frac{S}{a} \times \frac{2b-1}{2br} + \frac{2b-3}{2br^2} + \frac{2b-5}{2br^3} + \frac{2b-7}{2br^4}$, &c. for the whole value required. Where the series is to be continued

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continued to as many terms as there are units in b , the complement of the oldest life.

Now, seeing the conclusion here brought out is not otherwise affected by a than its being a general divisor to the *whole*, it is evident that the true value of the expectation, supposing b to remain the same, will be reciprocally as the complement (a) of the younger life, and therefore, also, as the expectation of its duration, which is the half thereof. But the said value, when $a=b$, is in the same proportion to the given sum S , as half the reversion of an annuity for ever after two such equal *joint lives*, is to the *perpetuity* (*Vid. Doct. Ann. p. 76.*) Whence the Reason of the solution to *Prob. 32* is manifest. And, by proceeding according to the very same method, the 33d will also appear conspicuous.

But as to the 34th, the exegesis will be something more complex; since we must first of all determine the probability that both A and B will be extinct at the end of any number of years, on the contingency of A 's dying first.

In order to which, let a and b be taken as above specified; and suppose that a sum of money S is to be received on the decease of B (whenever that happens) provided A is then extinct.

Now (as before) the probability of receiving the sum S , within the compass of any proposed year, must be considered in two parts, according to the chance of one, or both lives, failing in that year.

Thus the probability of receiving it the 2d, 3d, or 4th, &c. year, on the contingency of A 's being dead before the commencement of that year, will be defined by $\frac{1}{a} \times \frac{1}{b}$, $\frac{2}{a} \times \frac{1}{b}$, or $\frac{3}{a} \times \frac{1}{b}$, &c. respectively.

And the probability of receiving it any year, on the contingency of both the lives failing within that year, will constantly be expressed by $\frac{1}{2ab}$,
fo

for as long as they both have a chance of continuing together.

Hence the whole probability, depending on the 1st, 2d, 3d, 4th, &c. year (by adding $\frac{1}{2ab}$ to each of the values first found) appears to be $\frac{1}{2ab}$, $\frac{3}{2ab}$, $\frac{5}{2ab}$, $\frac{7}{2ab}$, &c. respectively: therefore the sum of all these continued to as many terms (n) as there are years proposed, will, evidently, be the true value of the probability required. But the sum of n terms of the series, 1, 3, 5, 7, &c. is $= n^2$: Therefore the required probability is also truly expressed by $\frac{n^2}{2ab}$.

But here we must not omit to take notice, that this conclusion holds only when (n) the number of years is less than the time that A and B have a chance of living together: For, if n be greater than a , and the age of A at the same time exceeds that of B; then, besides the probability of receiving the sum S , in the time (a) during which A hath a chance of living, there is a further chance or probability, depending on the contingency of receiving it, or of B's dying, after the expiration of the said time: which (as the life A is then out of the question) will consequently be expressed by $\frac{n-a}{b}$. And this added to $\frac{aa}{2ab}$,

found above (by writing a for n) gives $\frac{2n-a}{2b}$, for the true measure of the required probability in this case, where n is greater than a .

Now, to apply these conclusions to the problem in question, let c denote the complement of the life C: then, as the probability of his receiving the rent of any proposed year is compounded of the probability of his living till then, and of the probability that both A and B are extinct, with the restriction of A's dying first (as above determined); it is manifest that the present
I value

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value of his expectations depending on the 1st, 2^d, 3^d, 4th, &c. years income, will be expressed by $\frac{1^2 \times c - 1}{2abc}$,

$$\frac{2^2 \times c - 2}{2abc^2}, \quad \frac{3^2 \times c - 3}{2abc^3}, \quad \frac{4^2 \times c - 4}{2abc^4}, \quad \&c. \text{ respectively:}$$

and so the whole expectation, or the value sought, will be truly exhibited by the following series $\frac{1}{2abc} \times$

$$\frac{1^2 \times c - 1}{r} + \frac{2^2 \times c - 2}{r^2} + \frac{3^2 \times c - 3}{r^3} + \dots + \frac{c^2 \times c - c}{r^c},$$

when C (according to case 1 of the solution) is older

than either A or B; and by $\frac{1}{2abc} \times \frac{1^2 \times c - 1}{r} +$

$$\frac{2^2 \times c - 2}{r^2} + \dots + \frac{b^2 \times c - b}{r^b} + \frac{b^2 \times c - b - 1}{r^{b+1}} + \dots + \frac{b^2 \times c - c}{r^c},$$

when C is younger than B and older than A (according to the former part of case 2).

But the first of these two values is known to express half the value of the reversion of the life C after the longest of the lives A and B.

And with regard to the second, it is evident (because

$$\frac{1}{bbc} \times \frac{1^2 \times c - 1}{r} + \frac{2^2 \times c - 2}{r^2} + \dots + \frac{b^2 \times c - b}{r^b} + \frac{b^2 \times c - b - 1}{r^{b+1}} + \dots + \frac{b^2 \times c - c}{r^c} \text{ is the value of the reversion of the}$$

life C after the longest of two equal lives B, B) that the true value thereof must be equal to that of the said reversion multiplied by $\frac{b}{2a}$: Hence the former

part of case 2^o is also manifest. — As to the latter part, it is derived from thence by plain reasoning. For, since the absolute value of the reversion of the life C after the two given lives A and B, may be considered as composed of two parts, depending on the chance

chance of the older of the two lives surviving the younger, and on the chance of the younger's surviving the older; it is very easy to perceive that, if the former of these parts be taken from the whole, the other part will remain.

Thus much concerning the solution of our 34th problem, on which the three succeeding ones almost intirely depend; and therefore it will be unnecessary to dwell upon them here.

The solution to the 38th may however need some sort of explanation: In regard to which it is evident, that the probability of the expectant's receiving the proposed legacy (S) at the end of any particular year, is compounded of the probability of his continuing alive till then, and that of B's dying within the compass of the said year, on the contingency of A's being extinct before. Now the last of these is already

found to be $\frac{1}{2ab}$, $\frac{3}{2ab}$, $\frac{5}{2ab}$, &c. according as the 1st,

2d 3d, &c. year is assigned; whence it is plain that the expectation, on the contingency of coming into possession at the end of the first, second, third, &c.

year, will be $\frac{1 \times c - 1 \times S}{2abcr}$, $\frac{3 \times c - 2 \times S}{2abcr^2}$,

$\frac{5 \times c - 3 \times S}{2abcr^3}$, &c. respectively;

and, consequently that the sum of all these, or $\frac{S}{2bac} \times$

$$\frac{1 \times c - 1}{r} + \frac{3 \times c - 2}{r^2} + \frac{5 \times c - 3}{r^3} + \frac{7 \times c - 4}{r^4} + \text{&c.}$$

will be the whole value required.

But this series may be divided into two others,

viz. $\frac{S}{abc} \times \frac{1 \times c - 1}{r} + \frac{2 \times c - 2}{r^2} + \frac{3 \times c - 3}{r^3} + \frac{4 \times c - 4}{r^4} + \text{&c.}$

and $\frac{-S}{2abc} \times \frac{c-1}{r} + \frac{c-2}{r^2} + \frac{c-3}{r^3} + \frac{c-4}{r^4} + \frac{c-5}{r^5} + \text{&c.}$

I 2

And

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And the former of those may be again resolved into $\frac{S}{a} \times \frac{c-1}{cr} + \frac{c-2}{cr^2} + \frac{c-3}{cr^3} + \frac{c-4}{cr^4} + \frac{c-5}{cr^5} \&c.$

and $\frac{-S}{a} \times \frac{b-1 \times c-1}{bcr} + \frac{b-2 \times c-2}{bcr^2} + \frac{b-3 \times c-3}{bcr^3} \&c.$

whereof the last (exclusive of the general multiplicator) is known to express the value of the joint lives B and C: and the preceding part (or series) is likewise known to represent, either the value of the *whole* live C, or else the value of as many years of it as are expressed by the complement of the oldest of the lives A and B, according to the different cases specified in the solution of the problem: whence the reason of the whole is apparent.—As to the series $\frac{-S}{2abc}$

$\times \frac{c-1}{r} + \frac{c-2}{r^2} + \frac{c-3}{r^3} \&c.$ whereof no notice is taken

in the solution, it is so small in comparison of the other two, that to have embarrassed the solution with it would not only have been unnecessary, but quite improper, considering the nature of the subject; where an absolute exactness (for want of knowing the precise law of the decrements of life) is impossible.

There remain yet three other solutions to speak of; but as these depend intirely on the 38th and some of the preceding ones, it will be needless to say any thing further about them in this place. I have already been very particular on these kinds of problems, and the more so, as there is no method yet published (that I know of) by which they can be rightly determined. 'Tis true the manner of proceeding by first finding the probability of survivorship (which method is used in my former work, and which a celebrated author on this subject has largely insisted on in three successive editions) may be applied to good advantage when the given ages are nearly equal: but then,

then, it is certain that this is not a genuine way of going to work; and that the conclusions hence derived are at the best but near approximations. The rate of interest that money bears must be compounded with the probability of survivorship, and the expectation on each particular year must be determined, in order to have a true solution.

I shall conclude what I have to say on this subject of annuities with the solutions of two or three of the most useful problems, according to a method very different from that whereby they are usually investigated.

PROBLEM I.

To determine the present Value of an Estate in Land, for a single Life A of a given Age, according to the Hypothesis of an uniform decrease of the Probability of Life.

Let r denote the rate, p the perpetuity, and a the complement of the proposed life. Moreover let x (considered as flowing uniformly) be any time elapsed from his entering into possession: then the present value of the Expectation on the next succeeding

moment x , will evidently be expressed by $\frac{a-x}{a} \times$

$\frac{\dot{x}}{r^x}$, or its equal, $\dot{x} q^x - \frac{x \dot{x} q^x}{a}$ (by putting $q = \frac{1}{r}$)

whereof the fluent, when $x = a$, will consequently be the true value required.

But the fluent of the first term $(\dot{x} q^x)$ is found = $\frac{q^x}{m}$; and that of the second $\left(\frac{x \dot{x} q^x}{a}\right) = \frac{q^x}{ma} \times x - \frac{1}{m}$;

by Prob. 1, Sect. 6, Vol. 2, of my *Doctrine and Application of Fluxions*: where m denotes the Hyperbolical Logarithm of q .

These

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These fluents corrected by their constant quantities, become $\frac{q^x - 1}{m}$, and $\frac{q^x}{am} \times x - \frac{1}{m} + \frac{1}{am^2}$: which, when x is $= a$, are expressed by $\frac{q^a - 1}{m}$, and $\frac{q^a}{m} + \frac{1 - q^a}{am^2}$, respectively. The difference of which, or $-\frac{1}{m} - \frac{1 - q^a}{am^2}$, is consequently the true value of the proposed life.

But $\frac{q^x - 1}{m}$ (the fluent of $\dot{q}^x$), when x becomes infinite, will be equal to the perpetuity, that is $-\frac{1}{m}$ ($= \frac{1}{\text{Hyp. Log. } r}$) $= p$: whence, $-\frac{1}{m}$ being $= \frac{1}{p}$, the value found above will also be expressed by $p - \frac{1 - q^a \times p^2}{a}$, or its equal $p - \frac{pM}{a}$; supposing M to be put for $(1 - q^a \times p)$ the value certain for the time a .

The value here brought, it may be observed, exceeds $\left(p - \frac{p + 1 \times M}{a}\right)$ that given in p. 62, by $\frac{M}{a}$:

which is owing to the advantage of enjoying an estate in land (by an actual possession to the last moments of life) preferable to an annuity, where no regard is had to the time elapsed from the last yearly-payment to the dropping of the proposed life.

PROBLEM

PROBLEM II.

To determine the Value of an Estate in Land, to continue during the joint Existence of any Number of assigned Lives, A, B, C, D, &c. according to the aforesaid Hypothesis.

Let the complements of the proposed lives, A, B, C, D, &c. be denoted by a, b, c, d , &c. respectively; then supposing other things to remain as in the preceding problem, the value of the expectation on the moment x will here be expressed by $\frac{a-x}{a} \times \frac{b-x}{b} \times \frac{c-x}{c}$

$\times \frac{d-x}{d}$ &c. $\times \frac{x}{r^x}$, or its equal, $1 - \frac{x}{a} \times 1 - \frac{x}{b} \times 1 - \frac{x}{c}$ &c. q^x . Which, if the sum of all the quantities $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, &c. be put $= \alpha$; the sum of all their rectangles $= \beta$; the sum of all their solids $= \gamma$, &c. will be reduced to $xq^x - \alpha x^2 q^x + \beta x^3 q^x - \gamma x^4 q^x + \delta x^5 q^x$, &c.

Whereof the fluent (by the problem above quoted) will be found $= \frac{q^x}{m}$ into $1 - \alpha \times x - \frac{1}{m} + \beta \times x^2 - \frac{2x}{m} + \frac{2}{m^2}$

$- \gamma \times x^3 - \frac{3x^2}{m} + \frac{6x}{m^2} - \frac{6}{m^3} +$ &c. which, when x

is $= 0$, becomes $= \frac{1}{m} + \frac{\alpha}{m^2} + \frac{2\beta}{m^3} + \frac{6\gamma}{m^4} + \frac{24\delta}{m^5}$ &c.

Hence the corrected fluent (by subtracting this last expression, and writing p , instead of its equal —

$\frac{1}{m}$) will come out $p - \alpha p^2 + 2\beta p^3 - 6\gamma p^4 + 24\delta p^5$, &c.

$- q^x p$ into $1 - \alpha \times x + p + \beta \times x^2 + 2xp + 2p^2 - x \times x^3 + 3x^2 p + 6xp^2 + 6p^3$, &c. which, by taking x equal

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equal to the complement of the oldest life, gives the true value required.

When all the lives are equal, the conclusion will become much more simple: for the fluxion above given, by making $z=a-x$ ($=b-x=c-x$, &c.) will be here transformed to $-\frac{z^n r^z}{a^n r^a}$. Whose fluent, supposing m to denote the hyperbolical logarithm of r , will ap-

pear to be $-\frac{r^z}{ma^n r^a} \times z^n - \frac{nz^{n-1}}{m} + \frac{n \cdot n-1 \cdot z^{n-2}}{m^2}$ &c.

Which, when $x=0$, or $z=a$, becomes $-\frac{1}{ma^n} \times$

$a^n - \frac{na^{n-1}}{m} + \frac{n \cdot n-1 \cdot a^{n-2}}{m^2}$ &c. But when $x=a$,

or $z=0$, it becomes $-\frac{1}{ma^n r^a}$ into $+\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n}{m^n}$:

the difference of which values, or its equal,

$p - \frac{np^2}{a} + \frac{n \cdot n-1 \cdot p^3}{a^2} - \frac{n \cdot n-1 \cdot n-2 \cdot p^4}{a^3}$, &c.

$+\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n}{a^n r^a} \times p^n + 1$, is consequently the true

value of all the joint lives: where the sign of the last term must be negative, or positive, according as (n) the number of lives is even or odd.

PROBLEM III.

Suppose that a given Sum (S) is to be received as a Legacy, on the Decease of B, in case A is then extinct: To find its Value in present Money, allowing for that Contingency.

If a, b, q , &c. be supposed as in the preceding problems, it is easy to perceive that the Expectation on the moment x will here be defined by $\frac{x}{a} \times \frac{x}{b} \times$
 Sq^x

Sq^x , or its equal, $\frac{S}{a} \times \frac{x \dot{x} q^x}{b}$. But the fluent of $\frac{x \dot{x} q^x}{b}$ (when $x=b$) is known to express the Difference between the value of the life B, and the value (M) of an annuity certain for b years (*Vid. Prob. 1*). Whence it plainly follows, that $\frac{S}{a} \times \overline{M-B}$, will be the true value sought; provided B is older than A.

But, if B be younger than A; then, besides the expectation depending on the chance of his dying within the term (a) to the end of which A has a possibility of living, there is a farther expectation arising from the probability of B's not dying till after the expiration of the said term. Which expectation, with regard to any particular moment x (as A is here out of the question)

will be represented by $\frac{\dot{x}}{b} \times Sq^x$. Whose fluent (generated while x is increased from a to b) is expressed by $\frac{S}{b}$ drawn into the present value (V) of an annuity

certain for the time ($b-a$) which B has a chance of living after the greatest limit of A's duration. This,

therefore, added to $\frac{S}{b} \times \overline{N-A}$ (found as above, by changing a for b , A for B , and N for M) gives $\frac{S}{b} \times$

$\overline{N+V-A}$, for the whole value of the expectation in this case. But (N) the value of an annuity certain for the time a , with (V) the value of the reversion thereof for the time $b-a$, after the time a , is consequently equal to it's whole value (M) for the time

b . Therefore our last expression is reduced to $\frac{S}{b} \times \overline{M-A}$. From which and the other value found above, we have the following general rule.

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From

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From the value of an annuity in land, *certain* for as many years as are expressed by the complement of B's age, subtract the value of the oldest life (be it which it will) then say, as the complement of the youngest life, is to the remainder, so is the proposed sum to its required value in present money.

PROBLEM IV.

Q and his Heirs are intitled to an Estate of a given Value (S) on the Decease of C, in Case B survives A, and is himself survived by C. To determine the present Value of Q's Expectation.

Here the expectation on the moment x depends on the probability of C's dying in that moment, and on the probability that both A and B are before extinct, with the further restriction of A's dropping first.

In order to determine the latter of these, it will be proper to observe, from the last problem, that, as $\frac{x\dot{x}}{ab}$ denotes the probability of the expectant's receiving the sum S in the moment x , so $\left(\frac{x^2}{2ab}\right)$ the fluent thereof must consequently express the probability of receiving it during the interval x ; which is evidently the probability here required. Hence it follows that the expectation on the moment x , in the present case, is truly defined by $\frac{\dot{x}}{c} \times \frac{x^2}{2ab} S q^x$. Whose fluent, $\frac{Sq^x}{2abcm} \times \frac{x^2 - \frac{2x}{m} + \frac{2}{mm}}{}$, properly corrected, will consequently be the true value of Q's expectation.

After the same manner the value sought, in more complicated cases, may be determined. I shall here put down a general theorem for finding the value of an annuity, granted upon any number (n) of assigned lives, A, B, C, D, &c. but so, as to continue only,

only, as long as a given number (m) of them are in being.

Let Q be the value of all the joint lives, A, B, C, &c. that is, the value of an annuity, for as long as they all continue in being together; also let R be the sum of the values of all the joint lives that can arise, by combining A, B, C, &c. so as to leave out one life at each combination; and S the sum of all the joint lives that can arise by combining the same, so as to leave out two lives at each combination, &c. &c.

Then will the value of the purchase be truly expressed by $\pm \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} (n-m) \times Q \pm \frac{n-2}{1} \times \frac{n-3}{2} \times \frac{n-4}{3} (n-m-1) \times R \pm \frac{n-3}{1} \times \frac{n-4}{2} \times \frac{n-5}{3} (n-m-2) \times S \pm \frac{n-4}{1} \times \frac{n-5}{2} \times \frac{n-6}{3} (n-m-3) \times T, \&c.$

Where the upper or the lower signs obtain, according as $n-m$ is an even or an odd number; and where the quantities between the hooks, express the number of the preceding factors to be taken; with regard to which it is to be observed, that in the last term, where the number of factors becomes $= 0$, an unit must be taken.

This theorem (which is strictly true according to any law of the decrements of life) is the same in effect with that given at p. 26, of my Doctrine of Annuities, but rather more commodious.—Nevertheless, as the trouble of computing the values Q , R , S , &c. will still be very great, when several lives are concerned, the following approximation for the value of the longest life (as the annuity is usually held thereon) will also be found of use.

Let $a, b, c, d, e, f, \&c.$ express the values of any number of single lives; of which values let a be the greatest, b the next, and so on; also let m denote the interest and eight tenths of the interest of one pound for one year:

K 2

then

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then will the value of all the lives be nearly $= a + \frac{2 - mb \times b^2}{4a} + \frac{2 - mc \times c^3}{9ab} + \frac{2 - md \times d^4}{16abc} + \frac{2 - me \times e^5}{25abcd} + \text{&c.}$

Which theorem, in that case where all the lives are equal, will become much more simple, being then reducible to

$a + \frac{2 - ma \times a \times \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}, \text{ &c.}}{}$ Where as many terms of the series $\frac{1}{4} + \frac{1}{9} + \frac{1}{16}, \text{ &c.}$ are to be taken, as are expressed by the number of the lives, less one.

Of

OF INTEREST and ANNUITIES.

Interest may be either simple or compound; simple interest is that which is paid for the loan of any principal, or sum of money, lent out for some limited time, at a certain rate *per Cent.* agreed upon between the borrower and the lender, and is always proportional to the time. Thus, if the rate agreed upon be 4 *per Cent. per Annum*, or, which is the same, if the interest of 100*l.* for one year be 4*l.* then the simple interest of the same sum for two years, will be 8*l.* for three years 12*l.* for 4 years 16*l.* and so on for any other time in proportion.

Compound interest is that which arises by leaving the simple interest of any sum of money, after it becomes due, together with the principal, in the hands of the borrower, and thereby converting the whole into a new principal. Thus, he who lets out 100*l.* for one year, at the rate of 4 *per Cent.* has a right to receive 104*l.* at the year's end; which sum he may leave in the borrower's hands a second year as a new principal, in order to receive interest for the whole; and this interest (which will be found 4*l.* 3*s.* 2 $\frac{1}{3}$ *d.*) together with 4*l.* the interest of the first principal, for the first year, will be the compound interest of 100*l.* for two years: and so on, for any greater number of years. But I shall first give the investigation of the theorems for simple interest.

Let the rate *per Cent.* or the interest of 100*l.* for one year = r ; the months, weeks, or days in one year = t ; the months, weeks, or days which any sum, a , is lent out for = n ; and the amount of that sum, in the said time, *viz.* principal and interest, = b .

Then it will be as 100 is to r (the interest of 100*l.*) so is the proposed sum (a) to $\frac{ar}{100}$, the interest of that sum, for the same time. Again, as t , the time in which the

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the said interest is produced, is to n (the time proposed) so is $\frac{ar}{100}$, the interest in the former of these times, to $\frac{anr}{100t}$, that in the latter; which added to a , the principal, gives $a + \frac{anr}{100t} = b$ (the whole amount): from whence, we also have $a = \frac{100bt}{100t + nr}$, $r = \frac{100t \times \overline{b - a}}{an}$, and $n = \frac{100t \times \overline{b - a}}{ar}$: the use of which equations, or theorems, will appear by the following examples.

Examp. 1. What is the amount of 550*l.* at 4 *per Cent.* in seven months?

In this case we have $a=550$, $r=4$, $t=12$, $n=7$; and consequently $a + \frac{anr}{100t} = 550 + \frac{550 \times 7 \times 4}{100 \times 12} = 562\frac{1}{2}$ *l.* or 562*l.* 16*s.* 8*d.* the true value sought.

Examp. 2. What is the interest of 1*l.* for one day, at the rate of 5 *per Cent.*?

Here r being $=5$, $t=365$, $a=1$, and $n=1$, we have $\frac{anr}{100t} = \frac{5}{100 \times 365} = \frac{1}{100 \times 73} = 0.0001369863$, &c. = the decimal parts of a pound required.

Examp. 3. What sum, in ready money, is equivalent to 600*l.* due nine months hence, allowing 5 *per Cent.* discount?

Here r being $=5$, $t=12$, $n=9$, and $b=600$, we have a (by *Theorem 2*) $= \frac{100 \times 600 \times 12}{100 \times 12 + 9 \times 5} = 578.313$ *l.* or 578*l.* 6*s.* 3 $\frac{1}{4}$ *d.* which is the value required.

Examp. 4. At what rate of interest will 300*l.* in fifteen months, amount to, or raise a stock of 330*l.*?

In this case we have given $t=12$, $n=15$, $a=300$, and $b=330$; whence (by *Theorem 3*) r will come out $= \frac{100 \times 12 \times 30}{300 \times 15} = 8$; therefore 8 *per Cent.* is the rate required.

Examp.

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Examp. 5. In how many days will 365 *l.* at the rate of 4 *per Cent.* amount to, or raise a stock of 400 *l.*?

Here (by *Theorem 4*) we have $n = \frac{100 \times 365 \times 35}{365 \times 4}$
 $= 875 =$ the number of days required.

Of Annuities or Pensions in arrear, computed at simple Interest.

Annuities or pensions in arrear are such, which being payable or becoming due yearly, remain unpaid any number of years: and we are to compute what all those payments will amount to, allowing simple interest for their forbearance, from the time each particular payment becomes due; in order to which,

Let $\begin{cases} A = \text{the annuity, pension, or yearly rent.} \\ n = \text{the time, or number of years, it is forborn.} \\ r = \text{the interest of 1} \text{ } \textit{l.} \text{ for one year.} \\ m = \text{the amount of the annuity and its interest.} \end{cases}$

Then, as $1 : r :: A : rA$, the interest of the proposed sum or pension A for one year; which, as the last year's rent but one is forborn only one year, will express the whole interest of that rent, or payment: moreover, since the last year's rent but two is forborn two years, its interest will be $2rA$: and, in the same manner, that of the last year's rent but three, will appear to be $3rA$, &c. &c. whence it is manifest that the sum total of all these, or the whole interest to be received at the expiration of n years, for the forbearance of the proposed annuity or pension, will be truly defined by the arithmetical progression $rA + 2rA + 3rA + 4rA + 5rA$ &c. continued to $n - 1$ terms, that is, to as many terms as there are years, excepting the last. But the sum of this progression is equal to $n \times \frac{n-1}{2} \times rA$ (by *Theor. 4. Sect. 10.*)

Therefore, if to *this*, the aggregate of all the rents, or nA , be added, we shall have $nA + \frac{n \times n - 1}{2} \times rA = m$:
whence

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whence we also have $A = \frac{m}{n + n \times \frac{n-1}{2} \times r}$,

$r = \frac{2m - 2nA}{n \times n - 1 \times A}$, and $n = \sqrt{\frac{2m}{rA} + p^2} - p$; supposing

$$p = \frac{1}{r} - \frac{1}{2}.$$

Examp. 1. If 600*l.* yearly rent or pension be forborn five years, what will it amount to, allowing 4 per Cent. interest for each payment, from the time it becomes due?

Here we have given $A = 600$, $n = 5$, and $r = .04$ (for as $100 : 4 :: 1 : .04$) which values substituted in

Theorem 1. give $m = (nA + n \times \frac{n-1}{2} Ar = 3000 + 240) = 3240$ *l.* for the value that was to be found.

Examp. 2. What annuity or yearly pension, being forborn five years, will, in that time, amount to, or raise a stock of 3240*l.* at 4 per Cent. interest?

In this case we have given $n = 5$, $r = .04$, and $m = 3240$, and therefore, by *Theorem 2*, $A (= \frac{m}{n + \frac{1}{2}n \times n - 1 \times r} = \frac{3240}{5 + .4}) = 600$; which is the annuity required.

Examp. 3. At what rate of interest will an annuity of 560*l.* in seven years, raise a stock of 4508*l.*?

In this case we have given $A = 560$, $n = 7$, and $m = 4508$; whence (by *Theor. 3*) we have $r (= \frac{2m - 2nA}{n \times n - 1 \times A} = \frac{9016 - 7840}{42 \times 560} = .05$ = the interest of 1*l.* for one

year; therefore it will be as $1 : .05 :: 100 : 5$ (per Cent.) the rate required.

Examp. 4. How long must an annuity of 560*l.* be forborn, to raise a stock of 4508*l.* supposing interest to be 5 per Cent?

Here,

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Here we have given $A = 560$, $r = .05$, $m = 4508$; whence, by *Theorem 4*, we also have $p (= \frac{1}{r} - \frac{1}{2}) = 19.5$; and consequently $n (= \sqrt{\frac{2m}{rA} + p^2} - p) = 7$; which is the number required.

Note. If the rent or pension, is payable half-yearly, or quarterly, the method of proceeding will be *still* the same, provided n be always taken to express the number of payments, and r the interest of 1 *l.* for the time in which the first payment becomes due. Thus, if it were required to find what 300 *l.* half-yearly pension would amount to in five years at 4 *per Cent.* interest: then the simple interest of 1 *l.* for half a year being $= .02$, and the number of payments $= 10$, we, in this case have $A = 300$, $r = .02$, and $n = 10$; and consequently m (by *Theorem 1*) $= rA + n \times \frac{n-1}{2} \times rA = 3270$ *l.* which is the value sought. And the like is to be observed in what follows hereafter.

Of the present values of Annuities, or Pensions, computed at simple interest.

Let $\begin{cases} A = \text{the annuity, pension, or yearly rent.} \\ r = \text{the interest of 1} \text{ } l. \text{ for one year.} \\ n = \text{the number of years.} \\ v = \text{the present value of the annuity.} \end{cases}$

Then, because the amount of the annuity, in n years, is found above to be $nA + \frac{1}{2}n(n-1)rA$, and since 1 *l.* present money, is equivalent to $1 + nr$ to be received at the end of the time n , we therefore have $1 + nr : 1 :: nA + \frac{1}{2}n(n-1)rA$ (the said amount) $: \frac{nA + \frac{1}{2}n(n-1)rA}{1 + nr}$, its required value, in present

money. But it may be observed, that this method, given by authors for determining the values of annuities, according to *simple interest* is in reality a particular sort,

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or

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or species of *compound interest*: since the allowing of interest upon the annuity as it becomes due, is nothing less than allowing interest upon interest; the annuity itself being properly the simple interest, and the capital from whence it arises the principal. It is true the sum $1 + nr$, expressing the amount of 1*l.* is given, strictly speaking, according to *simple interest*: but the conclusion (as a late author (*) very justly observes) would be more congruous, and answer better, were the same allowances to be made therein, as are made in finding the amount of the annuity; that is, were interest upon interest to be taken once and no more. Agreeable to this assumption, r , the interest of 1*l.* being considered as an annuity, its amount in n years (by writing r for A , in the general formula above) will be given $= nr + \frac{1}{2}n \cdot n - 1 \cdot r^2$: to which the principal 1*l.* being added, the aggregate $1 + nr + \frac{1}{2}n \cdot n - 1 \cdot r^2$ will therefore be the whole amount of 1*l.* in the time n ; and so we shall have $1 + nr + \frac{1}{2}n \cdot n - 1 \cdot r^2 : 1 :: nA + \frac{1}{2}n \cdot n - 1 \cdot rA : \frac{2nA + n \cdot n - 1 \cdot rA}{2 + 2nr + n \cdot n - 1 \cdot r^2} = v$, the true

value of the annuity, according to the said hypothesis. From which equation others may be derived, by means whereof the different values of A , n , and r may be successively determined. But, as this method of allowing interest upon interest, once and no more, is arbitrary, and the valuation of annuities, according to simple interest, a matter of more speculation than real use, it being not only customary, but also most equitable to allow compound interest in these cases, I shall not stay to exemplify it, but proceed to

The Resolution of the various cases of compound interest, and of Annuities, as depending thereon.

Let $\begin{cases} R = \text{the amount of 1*l.* in one year, viz. principal and interest.} \\ P = \text{any sum put out at interest.} \end{cases}$

Let

* Mr. Hardy in his *Annuities*.

Let $\begin{cases} n = \text{the number of years it is lent for.} \\ a = \text{its amount in that time.} \\ A = \text{any annuity forborn } n \text{ years.} \\ m = \text{its amount.} \\ v = \begin{cases} \text{the present value of the annuity for the} \\ \text{same time.} \end{cases} \end{cases}$

Therefore, since one pound put out at interest in the first year is increased to R , it will be as 1 to R , so is R , the sum forborn the second year, to R^2 , the amount of one pound in two years; and therefore as 1 to R , so is R^2 , the sum forborn the third year, to R^3 , the amount in three years: whence it appears that R^n , or R raised to the power whose exponent is the number of years, will be the amount of one pound in those years. But as 1 is to its amount R^n , so is P to (a) its amount in the same time; whence we have $P \times R^n = a$. Moreover, because the amount of one pound in n years is R^n , its increase in that time will be $R^n - 1$; but its interest for one single year, or the annuity answering to that increase, is $R - 1$, therefore as $R - 1$ to $R^n - 1$,

so is A to m . Hence we get $\frac{A \times R^n - 1}{R - 1} = m$. Fur-

thermore, since it appears that one pound, ready money, is equivalent to R^n , to be received at the expiration of

n years, we have, as R^n to 1 , so is $\frac{A \times R^n - 1}{R - 1}$ (the sum in arrear) to v , its worth in ready money; and there-

$$\text{fore } \frac{A \times 1 - \frac{1}{R^n}}{R - 1} = v.$$

From which three original equations, others may be derived, by help whereof the various questions relating to compound interest, annuities in arrear, and the present values of annuities may be resolved.

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Thus, because PR^n is $= a$, there will come out $P = \frac{a}{R^n}$, and $R = \sqrt[n]{\frac{a}{P}}$, &c. or, by exhibiting the same equations in logarithms (which is the most easy for practice) we shall have

$$1^\circ \text{ Log. } a = \text{log. } P + n \times \text{log. } R.$$

$$2^\circ \text{ Log. } P = \text{log. } a - n \times \text{log. } R.$$

$$3^\circ \text{ Log. } R = \frac{\text{log. } a - \text{log. } P}{n}.$$

$$4^\circ n = \frac{\text{log. } a - \text{log. } P}{\text{log. } R}.$$

Which four theorems, or equations, serve for the four cases in compound interest.

Again, since m is $= \frac{A \times R^n - 1}{R - 1}$, we shall have

$$1^\circ \text{ Log. } m = \text{log. } A + \text{log. } R^n - \text{log. } R - 1.$$

$$2^\circ \text{ Log. } A = \text{log. } m - \text{log. } R^n + \text{log. } R + 1.$$

$$3^\circ n = \frac{\text{log. } mR - m + A - \text{log. } A}{\text{log. } R}.$$

$$4^\circ R^n - \frac{mR}{A} + \frac{m}{A} - 1 = 0.$$

To which the various questions relating to annuities in arrear are referred.

Moreover, seeing $A \times \frac{1 - \frac{1}{R^n}}{R - 1}$ is $= v$, we thence have,

$$1^\circ \text{ Log. } v = \text{log. } A + \text{log. } 1 - \frac{1}{R^n} - \text{log. } R - 1.$$

$$2^\circ \text{ Log. } A = \text{log. } v + \text{log. } R - 1 - \text{log. } 1 - \frac{1}{R^n}.$$

$$3^\circ n = \frac{\text{log. } A - \text{log. } A + v - vR}{\text{log. } R}.$$

$$4^\circ R^{n+1} - \frac{A}{v} + 1 \times R^n + \frac{A}{v} = 0.$$

The

The use of which theorems respecting the present values of annuities, as well as of the preceding ones for compound interest and annuities in arrear, will fully appear from the following examples.

Examp. 1. To find the amount of 575*l.* in seven years at four *per cent. per annum*, compound interest.

In this case we have given $P = 575$, $R = 1.04$, and $n = 7$; therefore, by *Theorem 1*, $\log. a = \log. 575 + 7 \log. 1.04 = 2.8789011$; and consequently $a = 756.66$, or 756*l.* 13*s.* 2½*d.* the value required.

Examp. 2. What principal put to interest, will raise a stock of 1000*l.* in fifteen years, at 5 *per Cent*?

Here, we have given $R = 1.05$, $n = 15$, and $a = 1000$; therefore, by *Theorem 2*, $\log. P = \log. 1000 - 15 \log. 1.05 = 2.6821605$; and consequently $P = 481.02$, or 481*l.* 0*s.* 4¾*d.* the value sought.

Examp. 3. In how long time will 575*l.* raise a stock of 756*l.* 13*s.* 2½*d.* at 4 *per Cent*?

In this case we have $R = 1.04$, $P = 575$, and $a = 756.66$; whence, by *Theor. 4*, $n = \frac{\log. 756.66 - \log. 575}{\log. 1.04} = 7$, the number of years required.

Examp. 4. To find at what rate of interest 481*l.* in fifteen years, will raise a stock of 1000*l.*

Here we have given $P = 481$, $a = 1000$, and $n = 15$; therefore, by *Theorem 3*, $\log. R = \frac{\log. 1000 - \log. 481}{15} = .0211903$, whence $R = 1.05$; consequently 5 *per Cent.* is the rate required.

The four last examples relate to the cases in compound interest; the four next are upon the forbearance of annuities.

Examp. 1. If 50*l.* yearly rent, or annuity, be forborn seven years, what will it amount to, at 4 *per Cent. per Annum*, compound interest?

Here we have $R = 1.04$, $A = 50$, and $n = 7$; and therefore,

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therefore, by *Theor. 1*, $\log. m (= \log. A + \log. R^n - 1 - \log. R - 1) = \log. 50 + \log. 1,04^7 - 1 - \log. 04 = 2,596597$; and consequently $m = 395$ l. the value that was to be found.

Examp. 2. What annuity, forborn seven years, will amount to, or raise a stock of 395l. at 4 per Cent. compound interest?

In this case we have given $R = 1,04$, $n = 7$, and $m = 395$; whence, by *Theorem 2*, $\log. A (= \log. m - \log. R^n - 1 + \log. R - 1) = \log. 395 - \log. 1,04^7 - 1 + \log. 04 = 1,6989700$; and consequently $A = 50$ l. which is the annuity required.

Examp. 3. In how long time will 50l. annuity raise a stock of 395l. at 4 per Cent. per Annum, compound interest.

Here, we have $R = 1,04$, $A = 50$, $m = 395$; and therefore, by *Theor. 3*, $n (= \frac{\log. mR - m + A - \log. A.}{\log. R}) = \frac{,1192559}{,0170333} = 7$, the number of years required.

Examp. 4. If 120l. annuity, forborn eight years, amounts to, or raises a stock of 1200l. what is the rate of interest?

In this case we have given $n = 8$, $A = 120$, and $m = 1200$, to find R ; therefore, by *Theorem 4*, we have $R^8 - 10R + 9 = 0$; from which, by any of the methods in Sect. 13, (*) the required value of R will be found $= 1,06287$; therefore the rate is 6,287, or 6l. 5s. 9d. per Cent. per Annum.

The solution of the last case, where the rate is required, being a little troublesome, I shall here put down an approximation (derived from the third general formula, at p. 165 (*)) which will be found to answer near the truth, provided the number of years is not very great.

Let

(*) *Treatise of Algebra.*

Let $Q = \frac{n \cdot n - 1 \cdot A}{2 \cdot m - n A}$; then will
 $\frac{3000 Q + 2 n - 1 \cdot 400}{6 Q \cdot 5 Q + 3 n - 4 + \frac{1}{8} \cdot n - 2 \cdot 11 n - 13}$ be the rate
 per Cent. required.

Thus, for example, let $n = 8$, $A = 120$, and $m = 1200$; then will $Q = \frac{56 \cdot 120}{2 \cdot 240} = 14$, and the rate itself
 $= \frac{42000 + 6000}{14 \times 90 + 75} = 6.287$, as above.

The preceding examples explain the different cases of annuities in arrear; in the following ones the rules for the valuation of annuities are illustrated.

Examp. 1. To find the present value of 100*l.* annuity, to continue seven years, allowing 4 per Cent. per Annum compound interest.

Here, we have given $R = 1.04$, $A = 100$, and $n = 7$; and therefore, by Theorem 1, $\log. v (= \log. A + \log. 1 - \frac{1}{R^n} - \log. R - 1) = \log. 100 + \log.$

$1 - \frac{1}{1.04^7} - \log. .04 = 2.778296$; and consequently
 $v = 600.2 = 600*l.* 4*s.*$ which is the value that was to be found.

Examp. 2. What annuity, or yearly income, to continue 20 years, may be purchased for 1000*l.* at $3\frac{1}{2}$ per Cent?

In this case, $R = 1.035$, $n = 20$, $v = 1000$; whence, by Theorem 2, we have $\log. A (= \log. v + \log. R - 1 - \log. 1 - \frac{1}{R^n}) = 1.847336$; and consequently $A = 70.36$, or 70*l.* 7*s.* 2*d.*

Examp. 3. For how long time may one, with 600*l.* purchase an annuity of 100*l.* at 4 per Cent.

In

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In this example we have $R = 1,04$, $A = 100$, and $v = 600$; and therefore, by *Theorem 3*, n ($= \frac{\log. A - \log. A + v - vR}{\log. R}$) $= 7$, the number of years required.

Examp. 4. To determine at what rate of interest, an annuity of 50*l.* to continue 10 years, may be purchased for 400*l.*

Here $A = 50$, $n = 10$, and $v = 400$; whence by *Theorem 4*, $R^{n+1} - \frac{A}{v} + 1 \times R^n + \frac{A}{v}$ being $= 0$, we have $R^{11} - 1,125 R^{10} + 1,125 = 0$; which equation resolved, gives the required value of $R = 1,042775$; and consequently the rate of interest, 4,2775*l. per Annum.*

The solution of this last case being somewhat tedious, the following approximation (which will be found to answer very near the truth when the number of years is not very large) may be of use.

Assume $Q = \frac{n \cdot n + 1 \cdot A}{2nA - 2v}$: so shall

$\frac{3000 Q - 2n + 1 \times 400}{6Q \cdot 5Q - 3n - 4 + \frac{1}{6} \cdot n + 2 \cdot 11n + 13}$ expresses the rate *per Cent.* very nearly.

Thus, for example, let A (as above) be $= 50$, $n = 10$, and $v = 400$; then, Q being $= \frac{10 \times 11 \times 50}{1000 - 800} = 27,5$, we have $\frac{82500 - 8400}{165 \times 103,5 + 246}$, or 4,2775, for the rate, *per Cent.* the same as before.

F I N I S.

E R R A T A.

Page 19, line 10, in notes, from bottom, for P. 2. read P. 6.
23, . 1, for pag. 6. read pag. 10.

